Chapter 1

Advanced Euclidean Geometry

1.1 Role of Euclidean Geometry in High-School Mathematics

If only because in one's "further" studies of mathematics, the results (i.e., *theorems*) of Euclidean geometry appear only infrequently, this subject has come under frequent scrutiny, especially over the past 50 years, and at various stages its very inclusion in a high-school mathematics curriculum has even been challenged. However, as long as we continue to regard as important the development of logical, deductive reasoning in high-school students, then Euclidean geometry provides as effective a vehicle as any in bringing forth this worthy objective.

The lofty position ascribed to deductive reasoning goes back to at least the Greeks, with Aristotle having laid down the basic foundations of such reasoning back in the 4th century B.C. At about this time Greek geometry started to flourish, and reached its zenith with the 13 books of Euclid. From this point forward, geometry (and arithmetic) was an obligatory component of one's education and served as a paradigm for deductive reasoning.

A well-known (but not well *enough* known!) anecdote describes former U.S. president Abraham Lincoln who, as a member of Congress, had nearly mastered the first six books of Euclid. By his own admission this was not a statement of any particular passion for geometry, but that such mastery gave him a decided edge over his counterparts is dialects and logical discourse.

Lincoln was not the only U.S. president to have given serious thought to Euclidean geometry. President James Garfield published a novel proof in 1876 of the Pythagorean theorem (see Exercise 3 on page 4).

As for the subject itself, it is my personal feeling that the logical arguments which connect the various theorems of geometry are every bit as fascinating as the theorems themselves!

So let's get on with it ... !

1.2 Triangle Geometry

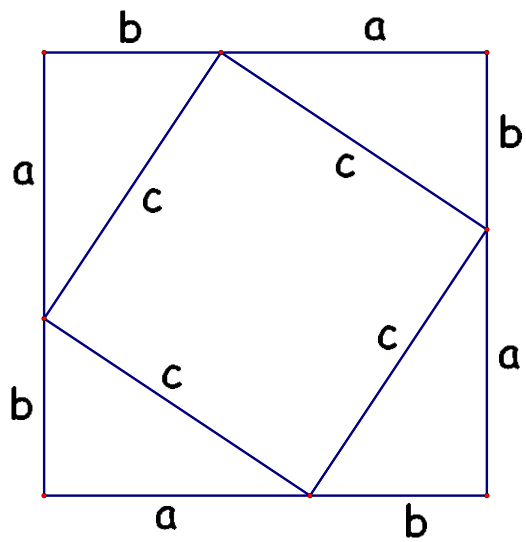
1.2.1 Basic notations

We shall gather together a few notational conventions and be reminded of a few simple results. Some of the notation is as follows:

A, B, C	labels of points
$[AB]$	The line segment joining A and B
AB	The length of the segment $[AB]$
(AB)	The line containing A and B
$\widehat{A}$	The angle at A
$C\widehat{A}B$	The angle between $[CA]$ and $[AB]$
$\triangle ABC$	The triangle with vertices A, B, and C
$\triangle ABC \cong \triangle A'B'C'$	The triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent
$\triangle ABC \sim \triangle A'B'C'$	The triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar

1.2.2 The Pythagorean theorem

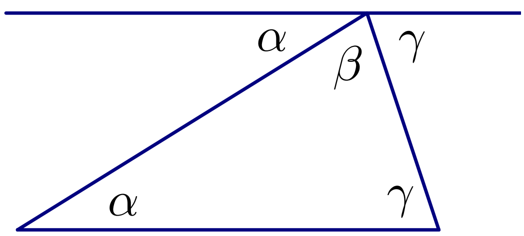
One of the most fundamental results is the well-known **Pythagorean Theorem**. This states that $\boldsymbol{a^2 + b^2 = c^2}$ in a right triangle with sides a and b and hypotenuse c. The figure to the right indicates one of the many known proofs of this fundamental result. Indeed, the area of the "big" square is $(a+b)^2$ and can be decomposed into the area of the smaller square plus the areas of the four congruent triangles. That is,



$$(a+b)^2 = c^2 + 2ab,$$

which immediately reduces to $a^2 + b^2 = c^2$.

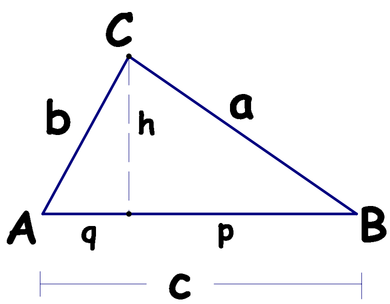
Next, we recall the equally well-known result that the sum of the interior angles of a triangle is 180°. The proof is easily inferred from the diagram to the right.



EXERCISES

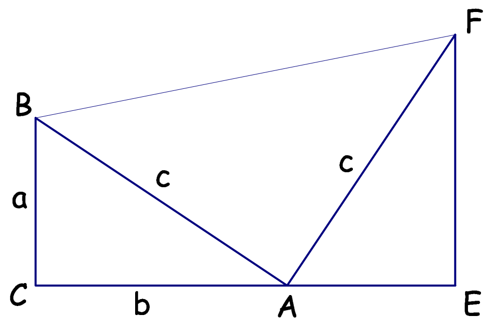
1. Prove **Euclid's Theorem for Proportional Segments**, i.e., given the right triangle $\triangle ABC$ as indicated, then

$$h^2 = pq, \quad a^2 = pc, \quad b^2 = qc.$$



2. Prove that the sum of the interior angles of a quadrilateral $ABCD$ is 360°.

3. In the diagram to the right, $\triangle ABC$ is a right triangle, segments $[AB]$ and $[AF]$ are perpendicular and equal in length, and $[EF]$ is perpendicular to $[CE]$. Set $a = BC$, $b = AB$, $c = AB$, and deduce President Garfield's proof[1] of the Pythagorean theorem by computing the area of the trapezoid $BCEF$.

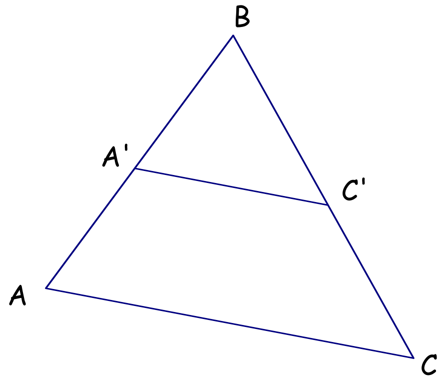


1.2.3 Similarity

In what follows, we'll see that many—if not most—of our results shall rely on the proportionality of sides in **similar triangles**. A convenient statement is as follows.

Similarity. Given the similar triangles $\triangle ABC \sim \triangle A'BC'$, we have that

$$\frac{A'B}{AB} = \frac{BC'}{BC} = \frac{A'C'}{AC}.$$



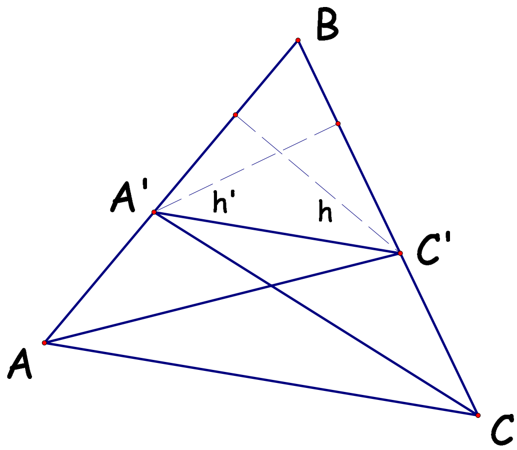
Conversely, if

$$\frac{A'B}{AB} = \frac{BC'}{BC} = \frac{A'C'}{AC},$$

then triangles $\triangle ABC \sim \triangle A'BC'$ are similar.

[1]James Abram Garfield (1831–1881) published this proof in 1876 in the JOURNAL OF EDUCATION (Volume 3 Issue 161) while a member of the House of Representatives. He was assasinated in 1881 by Charles Julius Guiteau. As an aside, notice that Garfield's diagram also provides a simple proof of the fact that perpendicular lines in the planes have slopes which are negative reciprocals.

PROOF. Note first that $\triangle AA'C'$ and $\triangle CA'C'$ clearly have the same areas, which implies that $\triangle ABC'$ and $\triangle CA'B$ have the same area (being the previous common area plus the area of the common triangle $\triangle A'BC'$). Therefore

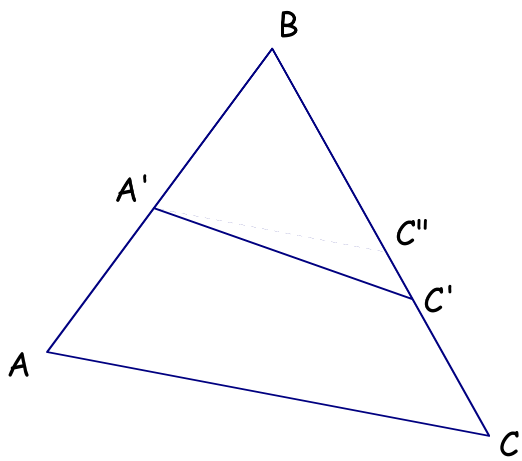


$$\begin{aligned} \frac{A'B}{AB} &= \frac{\frac{1}{2}h \cdot A'B}{\frac{1}{2}h \cdot AB} \\ &= \frac{\text{area}\,\triangle A'BC'}{\text{area}\,\triangle ABC'} \\ &= \frac{\text{area}\,\triangle A'BC'}{\text{area}\,\triangle CA'B} \\ &= \frac{\frac{1}{2}h' \cdot BC'}{\frac{1}{2}h' \cdot BC} \\ &= \frac{BC'}{BC} \end{aligned}$$

In an entirely similar fashion one can prove that $\dfrac{A'B}{AB} = \dfrac{A'C'}{AC}$. Conversely, assume that

$$\frac{A'B}{AB} = \frac{BC'}{BC}.$$

In the figure to the right, the point C'' has been located so that the segment $[A'C'']$ is parallel to $[AC]$. But then triangles $\triangle ABC$ and $\triangle A'BC''$ are similar, and so



$$\frac{BC''}{BC} = \frac{A'B}{AB} = \frac{BC'}{BC},$$

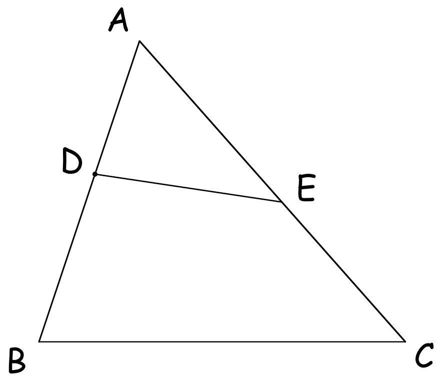
i.e., that $BC'' = BC'$. This clearly implies that $C' = C''$, and so $[A'C']$ is parallel to $[AC]$. From this it immediately follows that triangles

$\triangle ABC$ and $\triangle A'BC'$ are similar.

EXERCISES

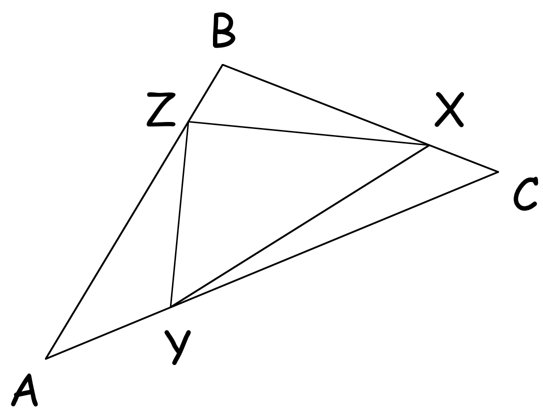
1. Let $\triangle ABC$ and $\triangle A'B'C'$ be given with $A\widehat{B}C = A'\widehat{B'}C'$ and $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC}$. Then $\triangle ABC \sim \triangle A'B'C'$.

2. In the figure to the right, $AD = rAB,\ AE = sAC$. Show that
$$\frac{\text{Area}\,\triangle ADE}{\text{Area}\,\triangle ABC} = rs.$$



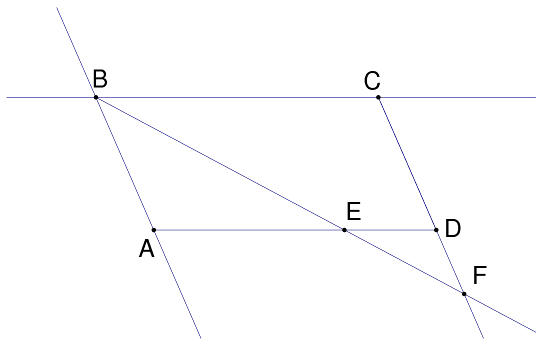
3. Let $\triangle ABC$ be a given triangle and let Y, Z be the midpoints of $[AC]$, $[AB]$, respectively. Show that (XY) is parallel with (AB). (This simple result is sometimes called the **Midpoint Theorem**)

4. In $\triangle ABC$, you are given that
$$\frac{AY}{YC} = \frac{CX}{XB} = \frac{BX}{ZA} = \frac{1}{x},$$
where x is a positive real number. Assuming that the area of $\triangle ABC$ is 1, compute the area of $\triangle XYZ$ as a function of x.

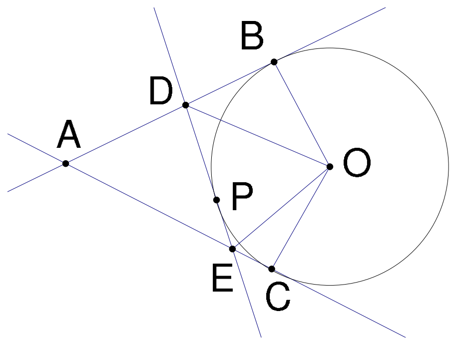


5. Let $ABCD$ be a quadrilateral and let $EFGH$ be the quadrilateral formed by connecting the midpoints of the sides of $ABCD$. Prove that $EFGH$ is a parallelogram.

6. In the figure to the right, $ABCD$ is a parallelogram, and E is a point on the segment $[AD]$. The point F is the intersection of lines (BE) and (CD). Prove that $AB \times FB = CF \times BE$.



7. In the figure to the right, tangents to the circle at B and C meet at the point A. A point P is located on the minor arc $\widehat{BC}$ and the tangent to the circle at P meets the lines (AB) and (AC) at the points D and E, respectively. Prove that $D\widehat{O}E = \frac{1}{2}B\widehat{O}C$, where O is the center of the given circle.



1.2.4 "Sensed" magnitudes; The Ceva and Menelaus theorems

In this subsection it will be convenient to consider the magnitude AB of the line segment $[AB]$ as "sensed,"[2] meaning that we shall regard AB as being either positive or negative and having absolute value equal to the usual magnitude of the line segment $[AB]$. The only requirement that we place on the signed magnitudes is that if the points A, B, and C are colinear, then

$$AB \times BC = \begin{cases} > 0 & \text{if } \overrightarrow{AB} \text{ and } \overrightarrow{BC} \text{ are in the same direction} \\ < 0 & \text{if } \overrightarrow{AB} \text{ and } \overrightarrow{BC} \text{ are in opposite directions.} \end{cases}$$

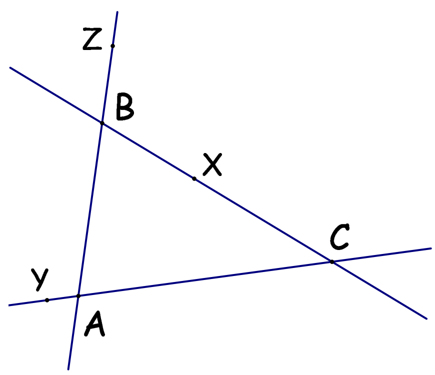
[2]IB uses the language "sensed" rather than the more customary "signed."

This implies in particular that for signed magnitudes,

$$\frac{AB}{BA} = -1.$$

Before proceeding further, the reader should pay special attention to the ubiquity of "dropping altitudes" as an auxiliary construction.

Both of the theorems of this subsection are concerned with the following configuration: we are given the triangle $\triangle ABC$ and points X, Y, and Z on the lines (BC), (AC), and (AB), respectively. Ceva's Theorem is concerned with the *concurrency* of the lines (AX), (BY), and (CZ). Menelaus' Theorem is concerned with the *colinearity* of the points X, Y, and Z. Therefore we may regard these theorems as being "dual" to each other.



In each case, the relevant quantity to consider shall be the product

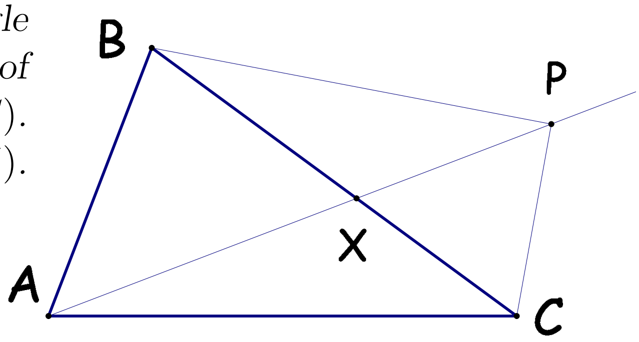
$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA}$$

Note that each of the factors above is nonnegative precisely when the points X, Y, and Z lie on the segments $[BC]$, $[AC]$, and $[AB]$, respectively.

The proof of Ceva's theorem will be greatly facilitated by the following lemma:

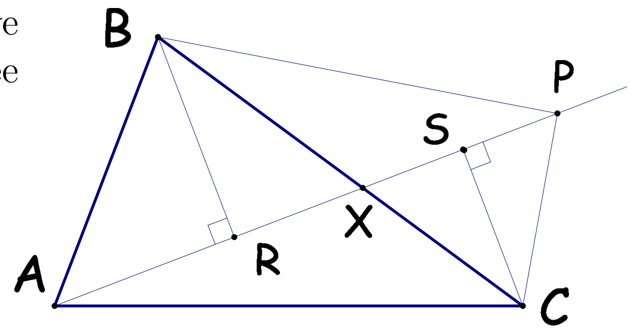
LEMMA. *Given the triangle $\triangle ABC$, let X be the intersection of a line through A and meeting (BC). Let P be any other point on (AX). Then*

$$\frac{\text{area } \triangle \text{APB}}{\text{area } \triangle \text{APC}} = \frac{BX}{CX}.$$



PROOF. In the diagram to the right, altitudes BR and CS have been constructed. From this, we see that

$$\begin{aligned} \frac{\text{area } \triangle \text{APB}}{\text{area } \triangle \text{APC}} &= \frac{\frac{1}{2}AP \cdot BR}{\frac{1}{2}AP \cdot CS} \\ &= \frac{BR}{CS} \\ &= \frac{BX}{CX}, \end{aligned}$$

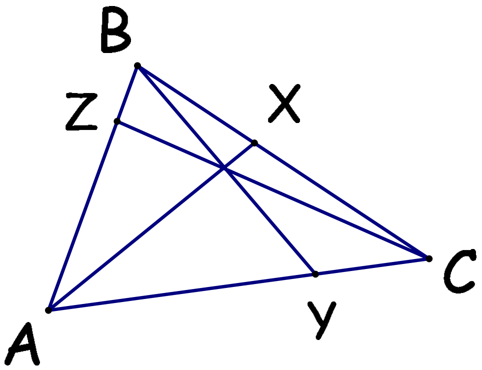


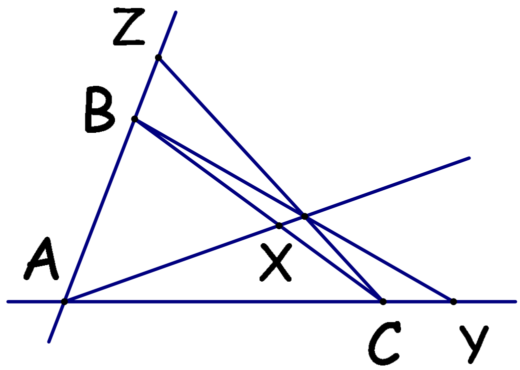
where the last equality follows from the obvious similarity $\triangle BRX \sim \triangle CSX$.

Note that the above proof doesn't depend on where the line (AP) intersects (BC), nor does it depend on the position of P relative to the line (BC), i.e., it can be on either side.

CEVA'S THEOREM. *Given the triangle $\triangle ABC$, lines (usually called* **Cevians** *are drawn from the vertices A, B, and C, with X, Y, and Z, being the points of intersections with the lines (BC), (AC), and (AB), respectively. Then (AX), (BY), and (CZ) are concurrent if and only if*

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = +1.$$





PROOF. Assume that the lines in question are concurrent, meeting in the point P. We then have, applying the above lemma three times, that

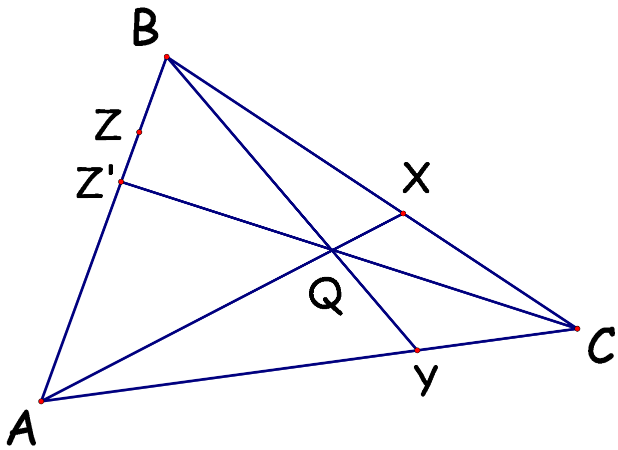
$$\begin{aligned} 1 &= \frac{\text{area}\,\triangle\text{APC}}{\text{area}\,\triangle\text{BPC}}\cdot\frac{\text{area}\,\triangle\text{APB}}{\text{area}\,\triangle\text{APC}}\cdot\frac{\text{area}\,\triangle\text{BPC}}{\text{area}\,\triangle\text{BPA}} \\ &= \frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}. \end{aligned}$$

To prove the converse we need to prove that the lines (AX), (BY), and (CZ) are concurrent, given that

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YZ} = 1.$$

Let $Q = (AX)\cap(BY)$, $Z' = (CQ)\cap(AB)$. Then (AX), (BY), and (CZ') are concurrent and so

$$\frac{AZ'}{Z'B}\cdot\frac{BX}{XC}\cdot\frac{CY}{YZ} = 1,$$

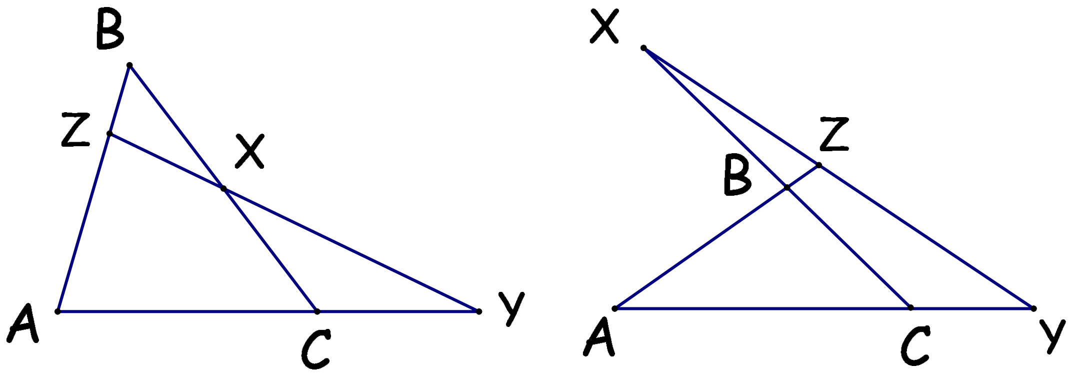


which forces

$$\frac{AZ'}{Z'B} = \frac{AZ}{ZB}.$$

This clearly implies that $Z = Z'$, proving that the original lines (AX), (BY), and (CZ) are concurrent.

Menelaus' theorem is a dual version of Ceva's theorem and concerns not **lines** (i.e., Cevians) but rather **points** on the (extended) edges of the triangle. When these three points are collinear, the line formed is called a **transversal**. The reader can quickly convince herself that there are two configurations related to $\triangle ABC$:



As with Ceva's theorem, the relevant quantity is the product of the sensed ratios:

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA};$$

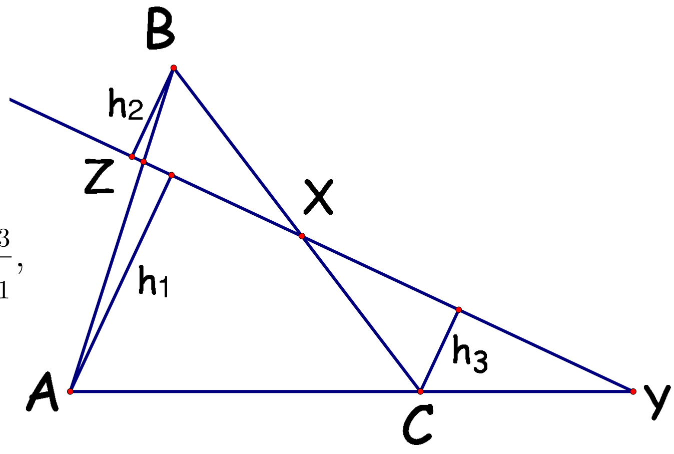
in this case, however, we see that either one or three of the ratios must be negative, corresponding to the two figures given above.

Menelaus' Theorem. *Given the triangle $\triangle ABC$ and given points X, Y, and Z on the lines (BC), (AC), and (AB), respectively, then X, Y, and Z are collinear if and only if*

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = -1.$$

Proof. As indicated above, there are two cases to consider. The first case is that in which two of the points X, Y, or Z are on the triangle's sides, and the second is that in which none of X, Y, or Z are on the triangle's sides. The proofs of these cases are formally identical, but for clarity's sake we consider them separately.

CASE 1. We assume first that X, Y, and Z are collinear and drop altitudes h_1, h_2, and h_3 as indicated in the figure to the right. Using obvious similar triangles, we get



$$\frac{AZ}{ZB} = +\frac{h_1}{h_2}; \quad \frac{BX}{XC} = +\frac{h_2}{h_3}; \quad \frac{CY}{YA} = -\frac{h_3}{h_1},$$

in which case we clearly obtain

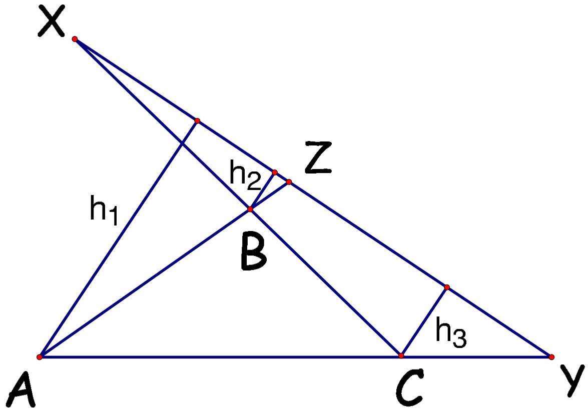
$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = -1.$$

To prove the converse, we may assume that X is on $[BC]$, Z is on $[AB]$, and that Y is on (AC) with $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$. We let X' be the intersection of (ZY) with $[BC]$ and infer from the above that

$$\frac{AZ}{ZB} \cdot \frac{BX'}{X'C} \cdot \frac{CY}{YA} = -1.$$

It follows that $\frac{BX}{XC} = \frac{BX'}{X'C}$, from which we infer easily that $X = X'$, and so X, Y, and Z are collinear.

CASE 2. Again, we drop altitudes from A, B, and C and use obvious similar triangles, to get



$$\frac{AZ}{ZB} = -\frac{h_1}{h_2}; \quad \frac{BX}{XC} = -\frac{h_2}{h_3}; \quad \frac{AY}{YC} = -\frac{h_1}{h_3};$$

it follows immediately that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1.$$

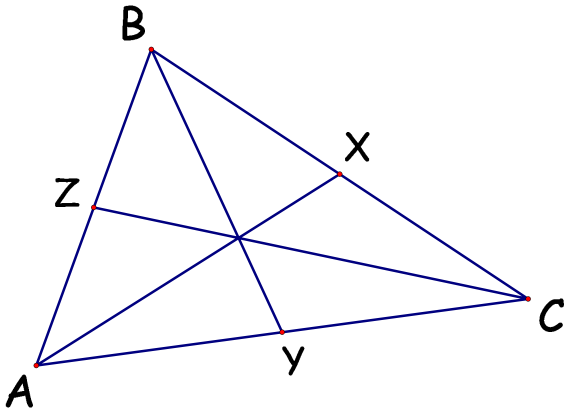
The converse is proved exactly as above.

1.2.5 Consequences of the Ceva and Menelaus theorems

As one typically learns in an elementary geometry class, there are several notions of "center" of a triangle. We shall review them here and show their relationships to Ceva's Theorem.

Centroid. In the triangle $\triangle ABC$ lines (AX), (BY), and (CZ) are drawn so that (AX) bisects $[BC]$, (BY) bisects $[CA]$, and (CZ) bisects $[AB]$ That the lines (AX), (BY), and (CZ) are concurrent immediately follows from Ceva's Theorem as one has that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YZ} = 1 \times 1 \times 1 = 1.$$



The point of concurrency is called the **centroid** of $\triangle ABC$. The three Cevians in this case are called **medians**.

Next, note that if we apply the Menelaus' theorem to the triangle $\triangle ACX$ and the transversal defined by the points B, Y and the centroid P, then we have that

$$1 = \frac{AY}{YC} \cdot \frac{CB}{BX} \cdot \frac{XP}{PA} \Rightarrow$$

$$1 = 1 \cdot 2 \cdot \frac{XP}{PA} \Rightarrow \frac{XP}{PA} = \frac{1}{2}.$$

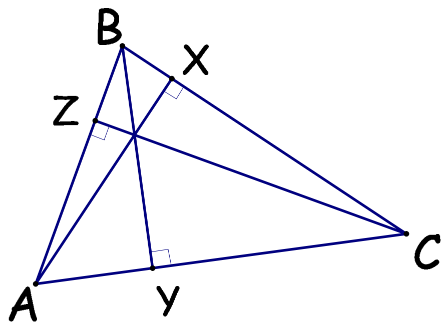
Therefore, we see that the distance of a triangle's vertex to the centroid is exactly 1/3 the length of the corresponding median.

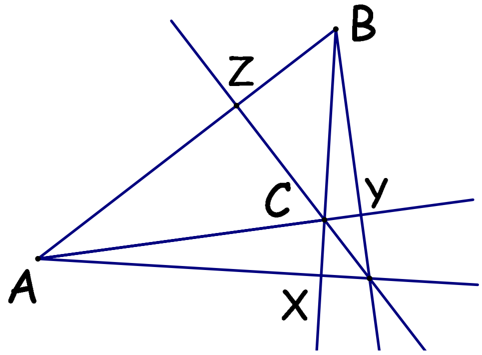
Orthocenter. In the triangle $\triangle ABC$ lines (AX), (BY), and (CZ) are drawn so that $(AX) \perp (BC)$, $(BY) \perp (CA)$, and $(CZ) \perp (AB)$. Clearly we either have

$$\frac{AZ}{ZB}, \frac{BX}{XC}, \frac{CY}{YA} > 0$$

or that exactly one of these ratios is positive. We have

$$\triangle ABY \sim \triangle ACZ \Rightarrow \frac{AZ}{AY} = \frac{CZ}{BY}.$$





Likewise, we have

$$\triangle ABX \sim \triangle CBZ \Rightarrow \frac{BX}{BZ} = \frac{AX}{CZ} \text{ and } \triangle CBY \sim \triangle CAX$$

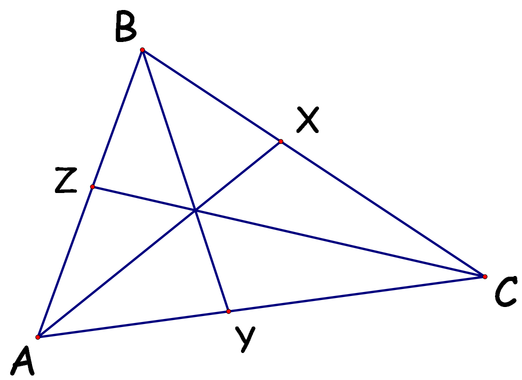
$$\Rightarrow \frac{CY}{CX} = \frac{BY}{AX}.$$

Therefore,

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AZ}{AY} \cdot \frac{BX}{BZ} \cdot \frac{CY}{CX} = \frac{CZ}{BY} \cdot \frac{AX}{CZ} \cdot \frac{BY}{AX} = 1.$$

By Ceva's theorem the lines (AX), (BY), and (CZ) are concurrent, and the point of concurrency is called the **orthocenter** of $\triangle ABC$. (The line segments $[AX]$, $[BY]$, and $[CZ]$ are the **altitudes** of $\triangle ABC$.)

Incenter. In the triangle $\triangle ABC$ lines (AX), (BY), and (CZ) are drawn so that (AX) bisects $B\widehat{A}C$, (BY) bisects $A\widehat{B}C$, and (CZ) bisects $B\widehat{C}A$ As we show below, that the lines (AX), (BY), and (CZ) are concurrent; the point of concurrency is called the **incenter** of $\triangle ABC$. (A very interesting "extremal"

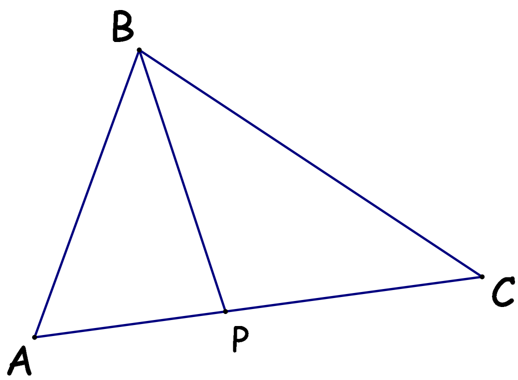


property of the incenter will be given in
Exercise 12 on page 153.) However, we shall proceed below to give another proof of this fact, based on Ceva's Theorem.

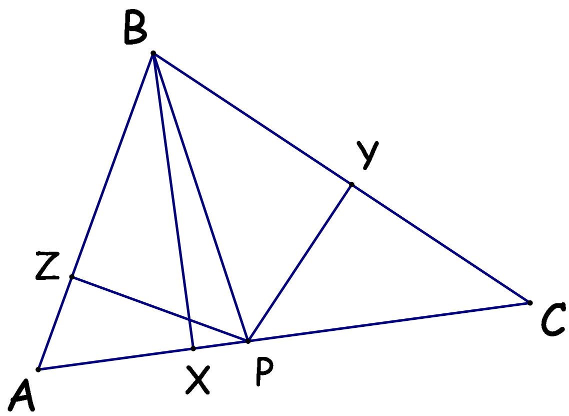
Proof that the angle bisectors of $\triangle ABC$ are concurrent. In order to accomplish this, we shall first prove the

Angle Bisector Theorem. *We are given the triangle $\triangle ABC$ with line segment $[BP]$ (as indicated to the right). Then*

$$\frac{AB}{BC} = \frac{AP}{PC} \Leftrightarrow A\widehat{B}P = P\widehat{B}C.$$



Proof ($\Leftarrow$). We drop altitudes from P to (AB) and (BC); call the points so determined Z and Y, respectively. Drop an altitude from B to (AC) and call the resulting point X. Clearly $PZ = PY$ as $\triangle PZB \cong \triangle PYB$. Next, we have



$$\triangle ABX \sim \triangle APZ \Rightarrow \frac{AB}{AP} = \frac{BX}{PZ} = \frac{BX}{PY}.$$

Likewise,

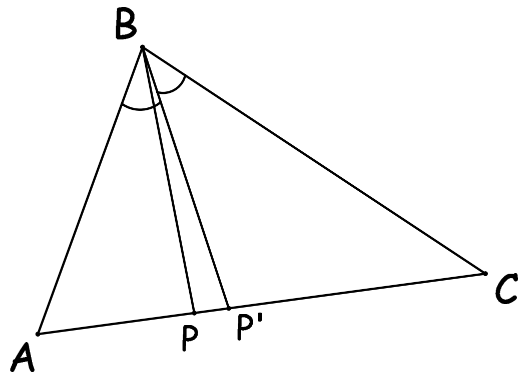
$$\triangle CBX \sim \triangle CPY \Rightarrow \frac{CB}{CP} = \frac{BX}{PY}.$$

Therefore,

$$\frac{AB}{BC} = \frac{AP \cdot BX}{PY} \cdot \frac{PY}{CP \cdot BX} = \frac{AP}{CP}.$$

(⇒). Here we're given that $\frac{AB}{BC} = \frac{AP}{PC}$. Let P' be the point determined by the angle bisector (BP') of $A\widehat{B}C$. Then by what has already been proved above, we have $\frac{AP}{BC} = \frac{AP'}{P'C}$. But this implies that

$$\frac{AP}{PC} = \frac{AP'}{P'C} \Rightarrow P = P'.$$



Conclusion of the proof that angle bisectors are concurrent. First of all, it is clear that the relevant ratios are all positive. By the Angle Bisector Theorem,

$$\frac{AB}{BC} = \frac{AY}{YC}, \ \frac{BC}{CA} = \frac{BZ}{ZA}, \ \frac{AB}{AC} = \frac{BX}{XC};$$

therefore,

$$\frac{AZ}{BZ} \times \frac{BX}{XC} \times \frac{CY}{YA} = \frac{CA}{BC} \times \frac{AB}{AC} \times \frac{BC}{AB} = 1.$$

Ceva's theorem now finishes the job!

EXERCISES

1. The Angle Bisector Theorem involved the bisection of one of the given triangle's **interior** angles. Now let P be a point on the line (AC) **external** to the segment $[AC]$. Show that the line (BP) bisects the external angle at B if and only if

$$\frac{AB}{BC} = \frac{AP}{PC}.$$

2. You are given the triangle $\triangle ABC$. Let X be the point of intersection of the bisector of $B\widehat{A}C$ with $[BC]$ and let Y be the point of intersection of the bisector of $C\widehat{B}A$ with $[AC]$. Finally, let Z be the point of intersection of the *exterior* angle bisector at C with the line (AB). Show that X, Y, and Z are colinear.[3]

[3] What happens if the exterior angle bisector at C is parallel with (AB)?

3. Given $\triangle ABC$ and assume that X is on (BC), Y is on (AC) and Z is on (AB). Assume that the Cevians $(AX)\,(BY)$, and (CZ) are concurrent, meeting at the point P. Show that

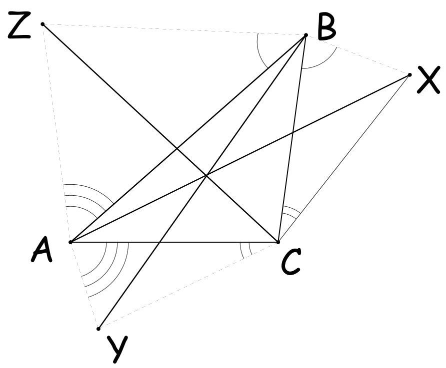
$$\frac{PX}{AX} + \frac{PY}{BY} + \frac{PZ}{CZ} = 1.$$

4. Given the triangle $\triangle ABC$ with incenter P, prove that there exists a circle $\mathcal{C}$ (called the **incircle** of $\triangle ABC$) with center P which is inscribed in the triangle $\triangle ABC$. The radius r of the incircle is often called the **inradius** of $\triangle ABC$.

5. Let $\triangle ABC$ have side lengths $a = BC,\ b = AC,$ and $c = AB$, and let r be the inradius. Show that the area of $\triangle ABC$ is equal to $\frac{r(a+b+c)}{2}$. (Hint: the incenter partitions the triangle into three smaller triangles; compute the areas of each of these.)

6. Given the triangle $\triangle ABC$. Show that the bisector of the **internal** angle bisector at A and the bisectors of the **external** angles at B and C are concurrent.

7. Given $\triangle ABC$ and points X, Y, and Z in the plane such that



$$\begin{aligned} \angle ABZ &= \angle CBX, \\ \angle BCX &= \angle ACY, \\ \angle BAZ &= \angle CAY. \end{aligned}$$

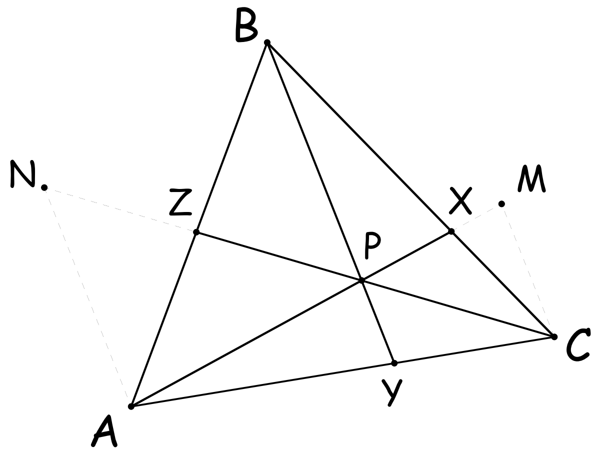
Show that (AX), (BY), and (CZ) are concurrent.

8. There is another notion of "center" of the triangle $\triangle ABC$. Namely, construct the lines l_1, l_2, and l_3 so as to be perpendicular bisectors of $[AB]$, $[BC]$, and $[CA]$, respectively. After noting that Ceva's theorem doesn't apply to this situation, prove directly that the lines l_1, l_2, and l_3 are concurrent. The point of concurrency is called the **circumcenter** of $\triangle ABC$. (Hint: argue that the point of concurrency of two of the perpendicular bisectors is equidistant to all three of the vertices.) If P is the circumcenter, then the common value $AP = BP = CP$ is called the **circumradius**

of the triangle $\triangle ABC$. (This is because the circumscribed circle containing A, B, and C will have radius AP.)

9. $\triangle ABC$ has side lengths $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are on sides $[AB]$ and $[AC]$, respectively such that $[DE] \parallel [BC]$ and $[DE]$ passes through the incenter of $\triangle ABC$. Compute DE.

10. Here's another proof of Ceva's theorem. You are given $\triangle ABC$ and concurrent Cevians $[AX]$, $[BY]$, and $[CZ]$, meeting at the point P. Construct the line segments $[AN]$ and $[CM]$, both parallel to the Cevian $[BY]$. Use similar triangles to conclude that



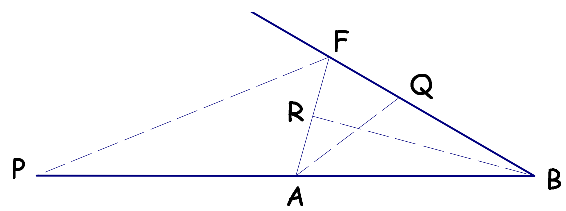
$$\frac{AY}{YC} = \frac{AN}{CM}, \frac{CX}{XB} = \frac{CM}{BP}, \frac{BZ}{ZA} = \frac{BP}{AN},$$

and hence that $\dfrac{AZ}{ZB} \cdot \dfrac{BX}{XC} \cdot \dfrac{CY}{YA} = 1.$

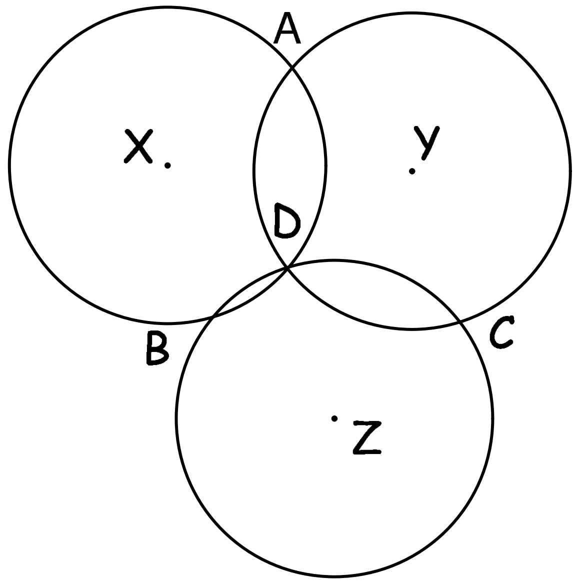
11. Through the vertices of the triangle $\triangle PQR$ lines are drawn lines which are parallel to the opposite sides of the triangle. Call the new triangle $\triangle ABC$. Prove that these two triangles have the same centroid.

12. Given the triangle $\triangle ABC$, let $\mathcal{C}$ be the inscribed circle, as in Exercise 4, above. Let X, Y, and Z be the points of tangency of $\mathcal{C}$ (on the sides $[BC]$, $[AC]$, $[AB]$, respectively) and show that the lines (AX), (BY), and (CZ) are concurrent. The point of concurrency is called the **Gergonne point** of the circle $\mathcal{C}$. (This is very easy once you note that $AZ = YZ$, etc.!)

13. In the figure to the right, the dotted segments represent angle bisectors. Show that the points P, R, and Q are colinear.



14. In the figure to the right, three circies of the same radius and centers X, Y and Z are shown intersecting at points A, B, C, and D, with D the common point of intersection of all three circles.
    Show that
    (a) D is the circumcenter of $\triangle XYZ$, and that
    (b) D is the orthocenter of $\triangle ABC$. (Hint: note that $YZCD$ is a rhombus.)



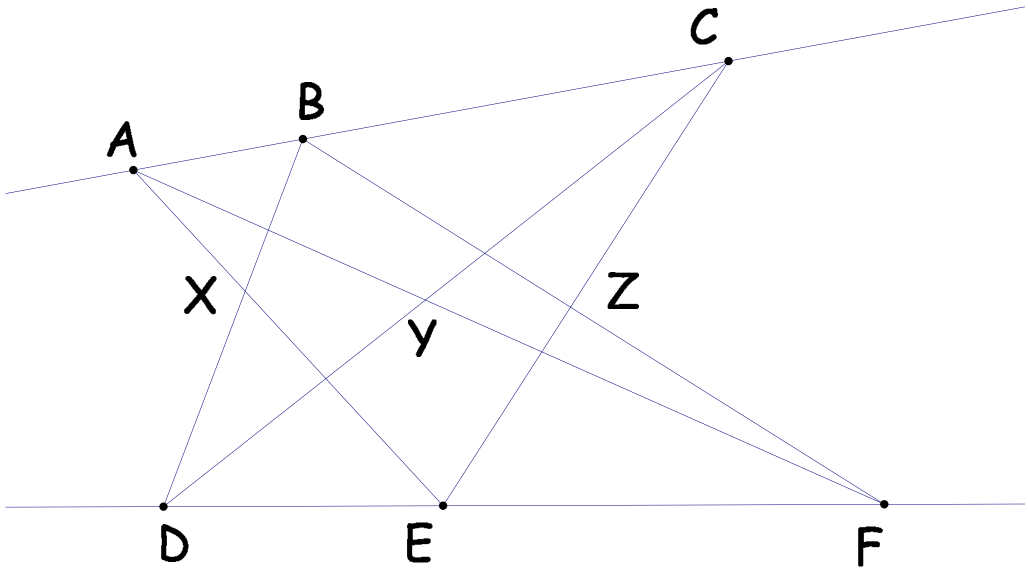
15. Show that the three medians of a triangle divide the triangle into six triangle of equal area.

16. Let the triangle $\triangle ABC$ be given, and let A' be the midpoint of $[BC]$, B' the midpoint of $[AC]$ and let C' be the midpoint of $[AB]$. Prove that

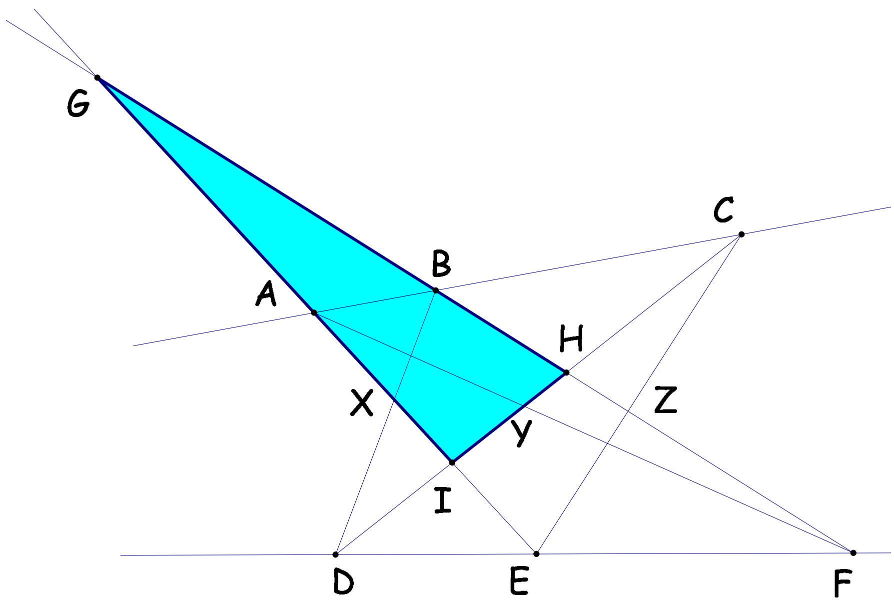
    (i) $\triangle A'B'C' \sim \triangle ABC$ and that the ratios of the corresponding sides are 1:2.
    (ii) $\triangle A'B'C'$ and $\triangle ABC$ have the same centroid.
    (iii) The four triangles determined within $\triangle ABC$ by $\triangle A'B'C'$ are all congruent.
    (iv) The circumcenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.

    The triangle $\triangle A'B'C'$ of $\triangle ABC$ formed above is called the **medial triangle** of $\triangle ABC$.

17. The figure below depicts a hexagram "inscribed" in two lines. Using the prompts given, show that the lines X, Y, and Z are colinear. This result is usually referred to **Pappus' theorem**.



Step 1. Locate the point G on the lines (AE) and (FB); we shall analyze the triangle $\triangle GHI$ as indicated below.[4]



Step 2. Look at the transversals, applying Menelaus' theorem to each:

[4]Of course, it may be that (AE) and (FB) are parallel. In fact, it may happen that all analogous choices for pairs of lines are parallel, which would render the present theme invalid. However, while the present approach uses Menelaus' theorem, which is based on "metrical" ideas, Pappus' theorem is a theorem only about incidence and colinearity, making it really a theorem belonging to "projective geometry." As such, if the lines (AE) and (BF) were parallel, then projectively they would meet "at infinity;" one could then apply a projective transformation to move this point at infinity to the finite plane, preserving the colinearity of X, Y, and Z

$[DXB]$, so $\dfrac{GX}{XI}\dfrac{ID}{DH}\dfrac{HB}{BG} = -1.$

$[AYF]$, so $\dfrac{GA}{AI}\dfrac{IY}{YH}\dfrac{HF}{FG} = -1.$

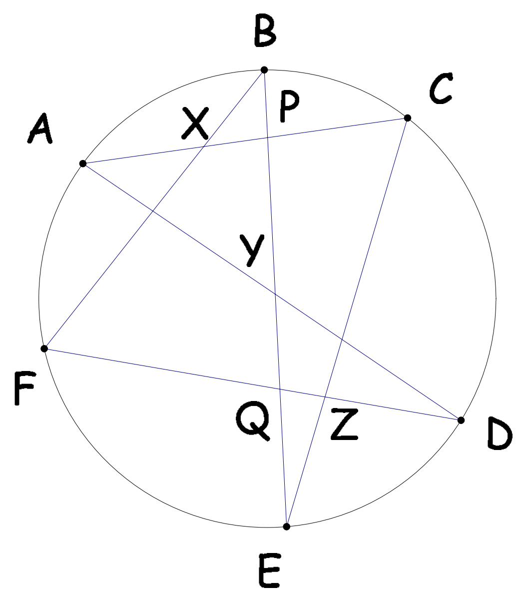
$[CZE]$ (etc.)

$[ABC]$ (etc.)

$[DEF]$ (etc)

Step 3. Multiply the above five factorizations of -1, cancelling out all like terms!

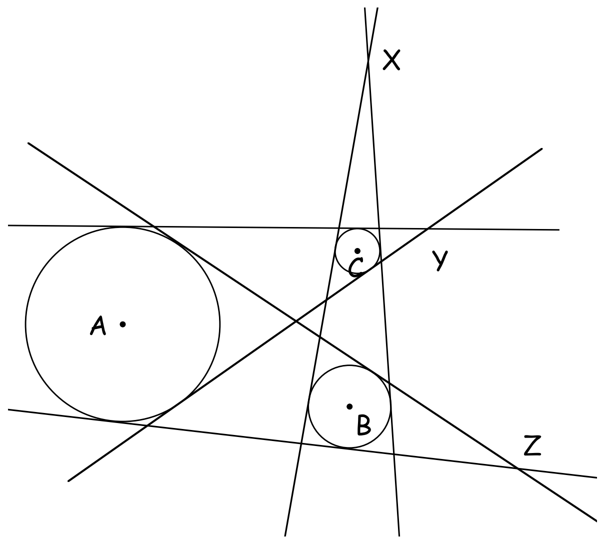
18. This time, let the hexagram be inscribed in a circle, as indicated to the right. By producing edges $[AC]$ and $[FD]$ to a common point R and considering the triangle $\triangle PQR$ prove **Pascal's theorem**, namely that points X, Y, and Z are colinear. (Proceed as in the proof of Pappus' theorem: consider the transversals $[BXF]$, $[AYD]$, and $[CZE]$, multiplying together the factorizations of -1 which each produces.)



19. A straight line meets the sides $[PQ]$, $[QR]$, $[RS]$, and $[SP]$ of the quadrilateral $PQRS$ at the points U, V, W, and X, respectively. Use Menelaus' theorem to show that

$$\frac{PU}{UQ} \times \frac{QV}{VR} \times \frac{RW}{WS} \times \frac{SX}{XP} = 1.$$

20. The diagram to the right shows three circles of different radii with centers A, B, and C. The points X, Y, and Z are defined by intersections of the tangents to the circles as indicated. Prove that X, Y, and Z are colinear.



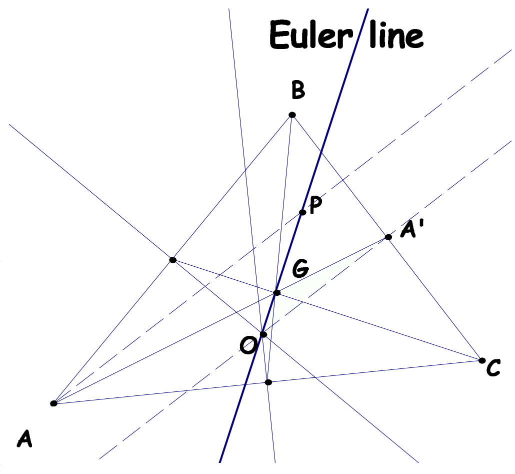
21. (**The Euler line.**) In this exercise you will be guided through the proof that in the triangle $\triangle ABC$, the centroid, circumcenter, and orthocenter are all colinear. The line so determined is called the **Euler line**.

    In the figure to the right, let G be the centroid of $\triangle ABC$, and let O be the circumcenter. Locate P on the ray $\overrightarrow{OG}$ so that $GP : OG = 2 : 1$.

    (a) Let A' be the intersection of (AG) with (BC); show that $\triangle OGA' \sim \triangle PGA$. (Hint: recall from page 13 that $GA : GA' = 2 : 1$.)

    (b) Conclude that (AP) and (OA') are parallel which puts P on the altitude through vertex A.

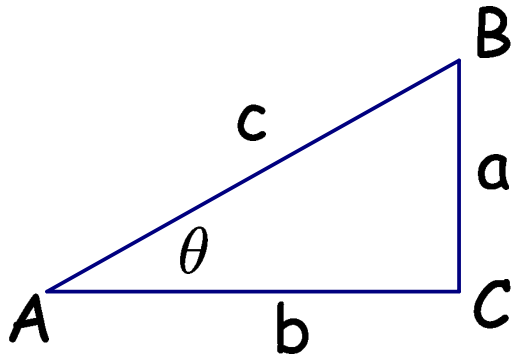
    (c) Similarly, show that P is also on the altitudes through vertices B and C, and so P is the orthocenter of $\triangle ABC$.



1.2.6 Brief interlude: laws of sines and cosines

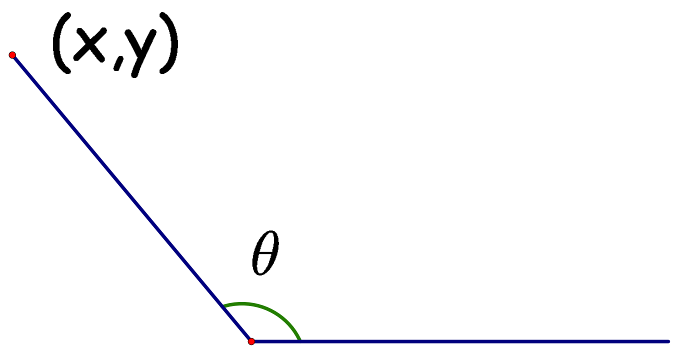
In a right triangle $\triangle ABC$, where $\widehat{C}$ is a right angle, we have the familiar **trigonometric ratios**: setting $\theta = \widehat{A}$, we have

$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c};$$



the remaining trigonometric ratios ($\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$) are all expressable in terms of $\sin\theta$ and $\cos\theta$ in the familiar way. **Of crucial importance here is the fact that by similar triangles, these ratios depend only on θ an not on the particular choices of side lengths.**[5]

We can extend the definitions of the trigonometric functions to arbitrary angles using coordinates in the plane. Thus, if θ is any given angle relative to the positive x-axis (whose measure can be anywhere between $-\infty$ and ∞ degrees, and if (x, y) is any point on the terminal ray, then we set



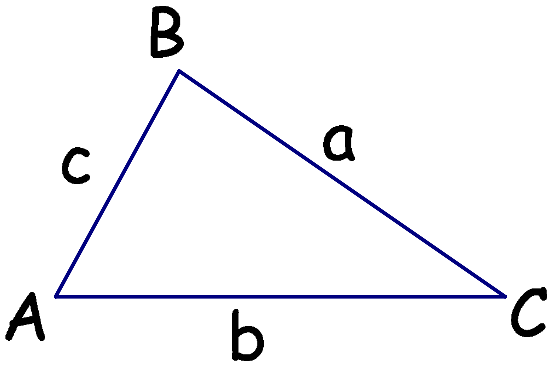
$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}.$$

Notice that on the basis of the above definition, it is obvious that $\sin(180-\theta) = \sin\theta$ and that $\cos(180-\theta) = -\cos\theta$. Equally important (and obvious!) is the **Pythagorean identity**: $\sin^2\theta + \cos^2\theta = 1$.

[5] A fancier way of expressing this is to say that by similar triangles, the trigonometric functions are **well defined**.

Law of Sines. *Given triangle $\triangle ABC$ and sides a, b, and c, as indicated, we have*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



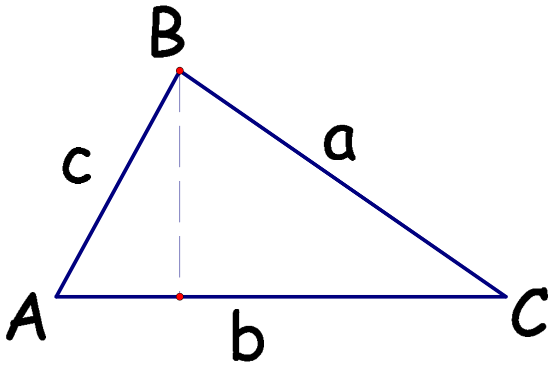
Proof. We note that

$$\frac{1}{2}bc\sin A = \text{area}\,\triangle ABC = \frac{1}{2}ba\sin C,$$

and so

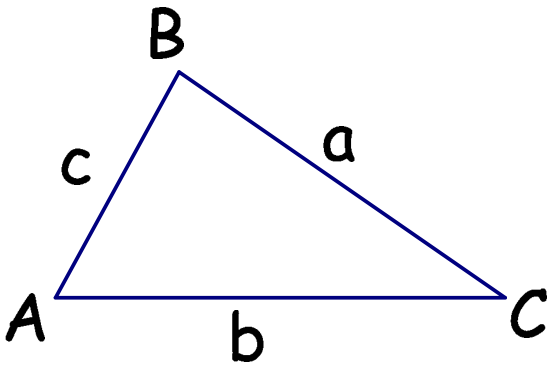
$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

A similar argument shows that $\dfrac{\sin B}{b}$ is also equal to the above.

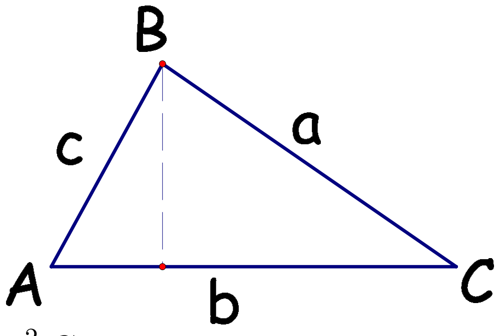


Law of Cosines. *Given triangle $\triangle ABC$ and sides a, b, and c, as indicated, we have*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$



PROOF. Referring to the diagram to the right and using the Pythagorean Theorem, we infer quickly that



$$\begin{aligned} c^2 &= (b - a\cos C)^2 + a^2\sin^2 C \\ &= b^2 - 2ab\cos C + a^2\cos^2 C + a^2\sin^2 C \\ &= a^2 + b^2 - 2ab\cos C, \end{aligned}$$

as required.

EXERCISES

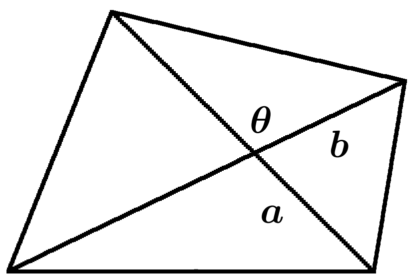
1. Using the Law of Sines, prove the Angle Bisector Theorem (see page 15).

2. Prove **Heron's formula**. Namely, for the triangle whose side lengths are a, b, and c, prove that the area is given by

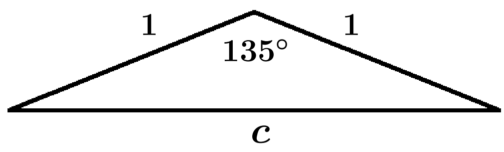
$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \dfrac{a+b+c}{2}$ = one-half the perimeter of the triangle. (Hint: if A is the area, then start with $16A^2 = 4b^2(c^2 - c^2\cos^2 A) = (2bc - 2bc\cos A)(2bc + 2bc\cos A)$. Now use the Law of Cosines to write $2bc\cos A$ in terms of a, b, and c and do a bit more algebra.)

3. In the quadrilateral depicted at the right, the lengths of the diagonals are a and b, and meet at an angle θ. Show that the area of this quadrilateral is $\frac{1}{2}ab\sin\theta$. (Hint: compute the area of each triangle, using the Law of Sines.)



4. In the triangle to the right, show that $c = \frac{\sqrt{1+i}+\sqrt{1-i}}{\sqrt[4]{2}}$ (where $i^2 = -1$)



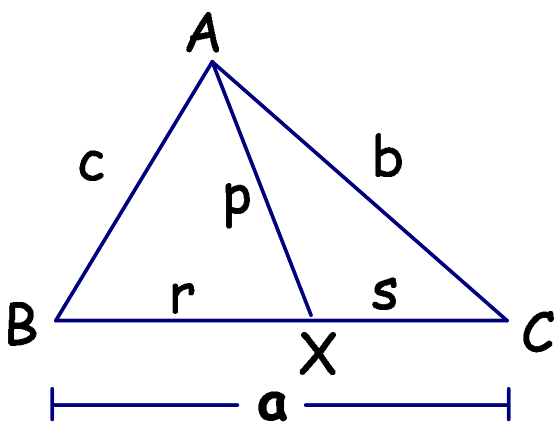
5. Given $\triangle ABC$ with C a right angle, let D be the midpoint of $[AB]$ and show that $\triangle ADC$ is isosceles with $AD = DC$.

6. Given $\triangle ABC$ with $BC = a$, $CA = b$, and $AB = c$. Let D be the midpoint of $[BC]$ and show that $AD = \frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$.

1.2.7 Algebraic results; Stewart's theorem and Apollonius' theorem

STEWART'S THEOREM. *We are given the triangle $\triangle ABC$, together with the edge BX, as indicated in the figure to the right. Then*

$$a(p^2 + rs) = b^2 r + c^2 s.$$



PROOF. We set $\theta = A\widehat{B}C$; applying the Law of Cosines to $\triangle AXB$ yields

$$\cos\theta = \frac{r^2 + p^2 - c^2}{2pr}.$$

Applying the Law of Cosines to the triangle $\triangle BXC$ gives

$$\cos\theta \ = \ \frac{b^2 - s^2 - p^2}{2ps}.$$

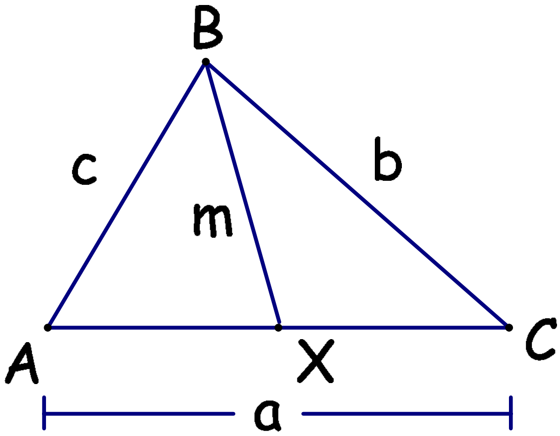
Equating the two expressions and noting that $a = r + s$ eventually leads to the desired result.

COROLLARY [APOLLONIUS THEOREM]. *We are given the triangle $\triangle ABC$, with sides a, b, and c, together with the median BX, as indicated in the figure to the right. Then*

$$b^2 + c^2 = 2m^2 + a^2/2.$$

If $b = c$ (the triangle is isosceles), then the above reduces to

$$m^2 + (a/2)^2 = b^2.$$



This follows instantly from Stewart's Theorem.

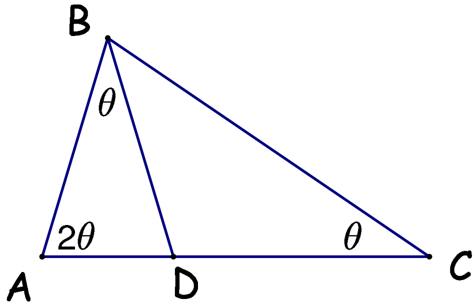
EXERCISES

1. Assume that the sides of a triangle are 4, 5, and 6.

   (a) Compute the area of this triangle.

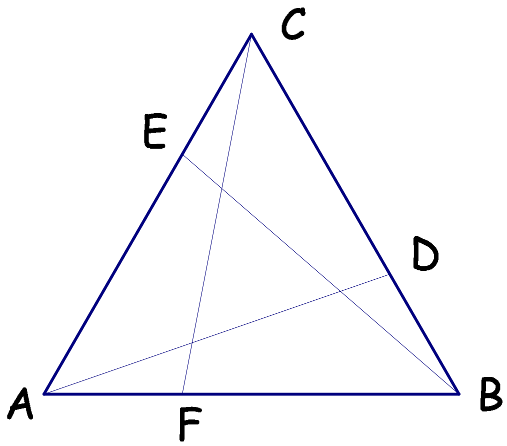
   (b) Show that one of the angles is twice one of the other angles.

2. **(The Golden Triangle)** You are given the triangle depicted to the right with $\triangle ABD \sim \triangle BCA$ Show that $\dfrac{DC}{AD} = \dfrac{\sqrt{5}+1}{2}$, the **golden ratio**.



3. Let $\triangle ABC$ be given with sides $a = 11$, $b = 8$, and $c = 8$. Assume that D and E are on side $[BC]$ such that $[AD]$, $[AE]$ trisect $B\widehat{A}C$. Show that $AD = AE = 6$.

4. You are given the equilateral triangle with sides of unit length, depicted to the right. Assume also that $AF = BD = CE = r$ for some positive $r < 1$. Compute the area of the inner equilateral triangle. (Hint: try using similar triangles and Stewart's theorem to compute $AD = BE = CF$.)

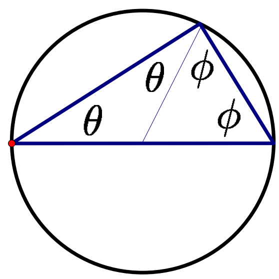


1.3 Circle Geometry

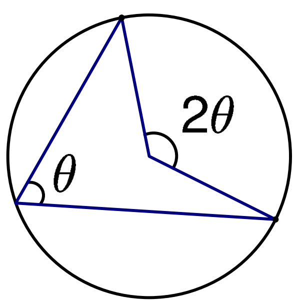
1.3.1 Inscribed angles

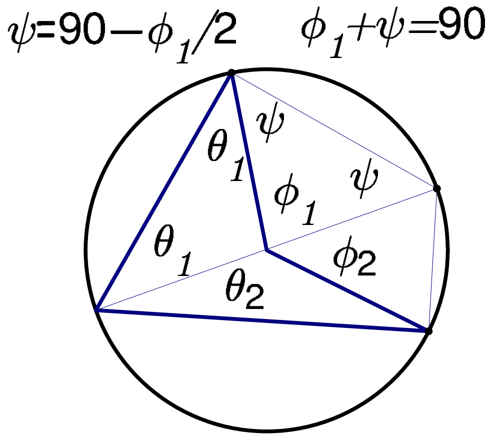
LEMMA. *If a triangle $\triangle ABC$ is inscribed in a circle with $[AB]$ being a diameter, then $A\widehat{C}B$ is a right angle.*

PROOF. The diagram to the right makes this obvious; from $2\theta + 2\phi = 180$, we get $\theta + \phi = 90°$.

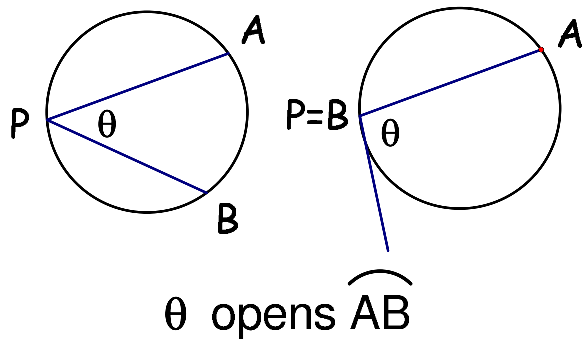


INSCRIBED ANGLE THEOREM. *The measure of an angle inscribed in a circle is one-half that of the inscribed arc.*



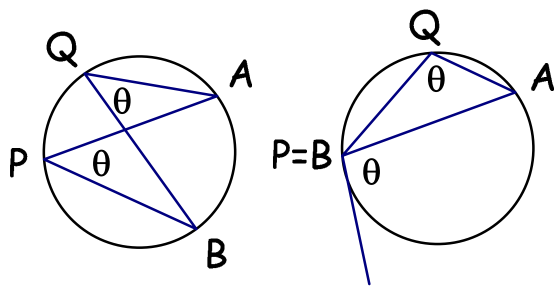


PROOF. We draw a diameter, as indicated; from the above lemma, we see that $\theta_1 + \psi = 90$. This quickly leads to $\phi_1 = 2\theta_1$. Similarly $\phi_2 = 2\theta_2$, and we're done.



Before proceeding, we shall find the following concept useful. We are given a circle and points A, B, and P on the circle, as indicated to the right. We shall say that the angle $A\widehat{P}B$ **opens** the arc $\widehat{AB}$. A degenerate instance of this is when B and P agree, in which case a tangent occurs. In this case we shall continue to say that the given angle **opens** the arc $\widehat{AB}$.

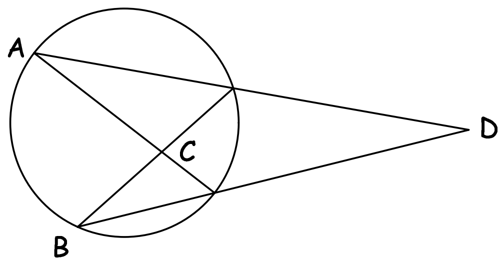
As an immediate corollary to the Inscribed Angle Theorem, we get the following:



COROLLARY. *Two angles which open the same are are equal.*

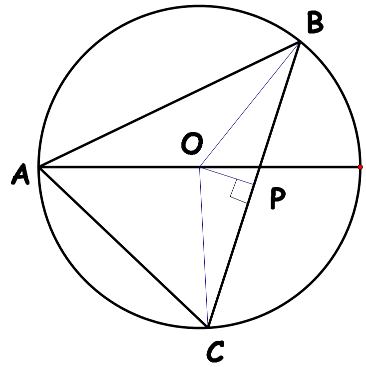
Exercises

1. In the diagram to the right, the arc $\widehat{AB}$ has a measure of 110° and the measure of the angle $A\widehat{C}B$ is 70°. Compute the measure of $A\widehat{D}B$.[6]

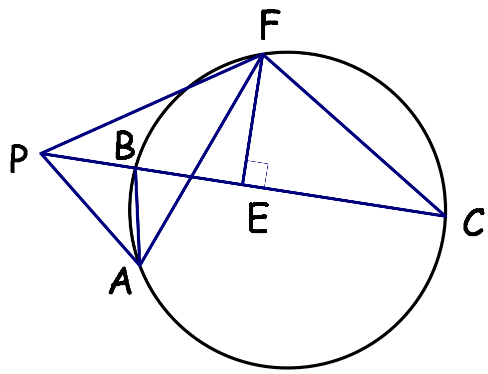


2. Let $[AB]$ be a diameter of the circle $\mathcal{C}$ and assume that C is a given point. If $A\widehat{C}B$ is a right angle, then C is on the circle $\mathcal{C}$.

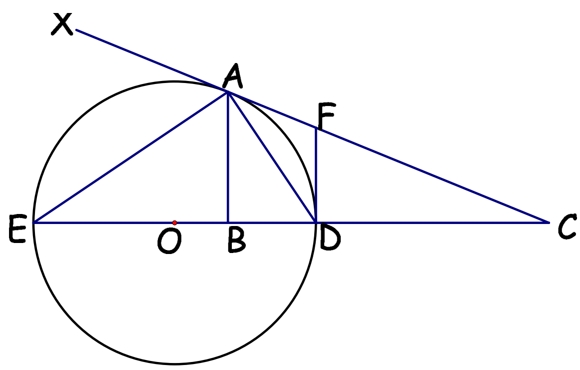
3. Let $\mathcal{C}$ be a circle having center O and diameter d, and let A, B, and C be points on the circle. If we set $\alpha = B\widehat{A}C$, then $\sin\alpha = BC/d$. (Hint: note that by the inscribed angle theorem, $B\widehat{A}C = P\widehat{O}C$. What is the sine of $P\widehat{O}C$?)



4. In the given figure $AF = FC$ and $PE = EC$.
   (a) Prove that triangle $\triangle FPA$ is isosceles.
   (b) Prove that $AB + BE = EC$.



5. A circle is given with center O. The points E, O, B, D, and E are colinear, as are X, A, F, and C. The lines (XC) and (FD) are tangent to the circle at the points A and D respectively. Show that
   (a) (AD) bisects $B\widehat{A}C$;
   (b) (AE) bisects $B\widehat{A}X$.



6. Let $\triangle ABC$ have circumradius R. Show that

$$\text{Area}\,\triangle ABC = \frac{R(a\cos A + b\cos B + c\cos B)}{2},$$

where $a = BC,\ b = AC$, and $c = AB$. (See exercise 5, page 17 for the corresponding result for the inscribed circle.)

Circle of Apollonius

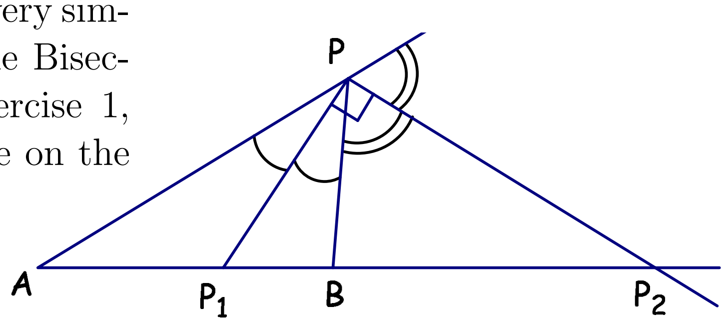
Circle of Apollonius. *Assume that $c \neq 1$ is a constant and that A and B are two given points. Then the locus of points*

$$\left\{P \mid \frac{PA}{PB} = c\right\}$$

is a circle.

Proof. This is actually a very simple application of the Angle Bisector Theorem (see also Exercise 1, page 16). Let P_1 and P_2 lie on the line (AB) subject to

$$\frac{AP_1}{P_1B} = c = \frac{AP_2}{BP_2}.$$



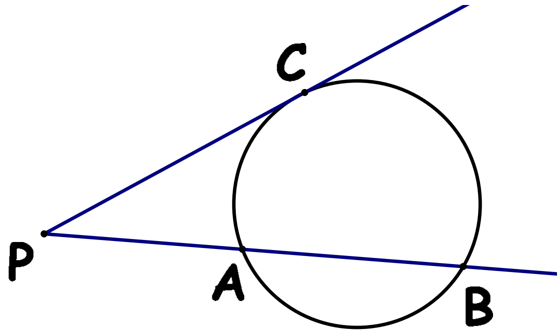
If we let P an arbitrary point also subject to the same condition, then from the Angle Bisector Theorem we infer that $A\widehat{P}P_1 = P_1\widehat{P}B$ and $B\widehat{P}P_2 = 180 - A\widehat{P}B$.

This instantly implies that $P_1\widehat{P}P_2$ is a right angle, from which we conclude (from Exercise 2, page 30 above) that P sits on the circle with diameter $[P_1P_2]$, proving the result.

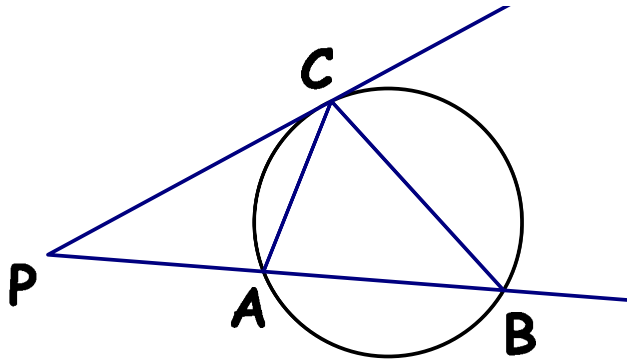
1.3.2 Steiner's theorem and the power of a point

SECANT-TANGENT THEOREM. *We are given the a circle, a tangent line (PC) and a secant line (PA), where C is the point of tangency and where $[AB]$ is a chord of the circle on the secent (see the figure to the right. Then*

$$PC^2 = PA \times PB.$$



PROOF. This is almost trivial; simply note that $P\widehat{C}A$ and $A\widehat{B}C$ open the same angle. Therefore, $\triangle PCA \sim \triangle PBC$, from which the conclusion follows.



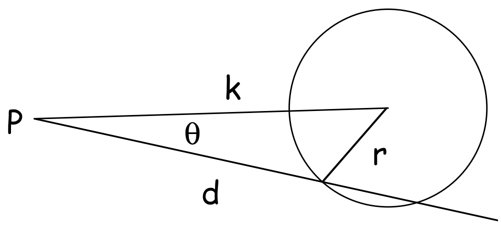
There is also an almost purely algebraic proof of this result.[7]

The following is immediate.

[7] If the radius of the circle is r and if the distance from P to the center of the circle is k, then denoting d the distance along the line segment to the two points of intersection with the circle and using the Law of Cosines, we have that $r^2 = k^2 + d^2 - 2kd\cos\theta$ and so d satisfies the quadratic equation

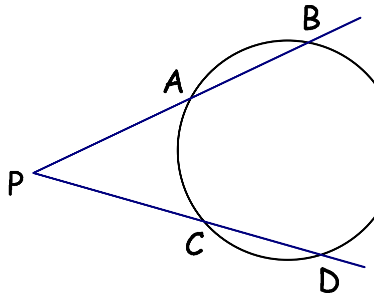
$$d^2 - 2kd\cos\theta + k^2 - r^2 = 0.$$

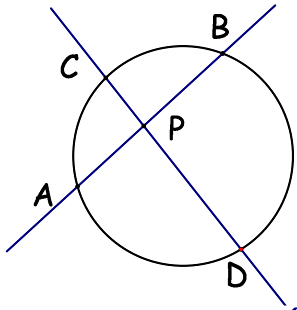
The product of the two roots of this equation is $k^2 - d^2$, which is independent of the indicated angle θ.



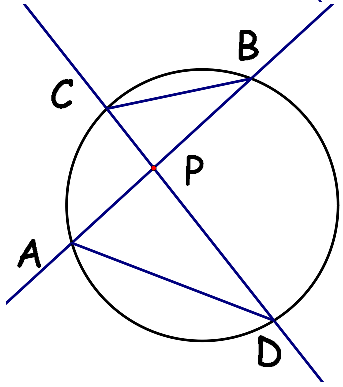
Corollary. **(Steiner's Theorem)** *We are given the a circle, and secant lines* (PA) *and* (PC)*, where* (PA) *also intersects the circle at* B *and where* (PC) *also intersects the circle at* D*.*

$$PA \times PB \;=\; PC \times PD.$$





Proof. Note that only the case in which P is interior to the circle needs proof. However, since angles $C\widehat{B}P$ and $P\widehat{D}A$ open the same are, they are equal. Therefore, it follows instantly that $\triangle PDA \sim \triangle PBC$, from which the result follows.



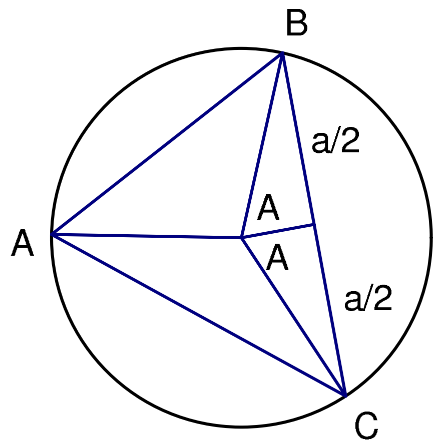
The product $PA \times PB$ of the distances from the point P to the points of intersection of the line through P with the given circle is independent of the line; it is called the **power of the point with respect to the circle**. It is customary to use *signed magnitudes* here, so that the power of the point with respect to the circle will be *negative* precisely when P is *inside* the circle. Note also that the power of the point P relative to a given circle $\mathcal{C}$ is a function only of the distance from P to the center of $\mathcal{C}$. (Can you see why?)

The second case of Steiner's theorem is sometimes called the "Intersecting Chords Theorem."

Exercises

1. In the complex plane, graph the equation $|z + 16| = 4|z + 1|$. How does this problem relate with any of the above?

2. Prove the "Explicit Law of Sines," namely that if we are given the triangle $\triangle ABC$ with sides a, b, and c, and if R is the circumradius, then



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Conclude that the perimeter of the triangle is
$a+b+c = 2R(\sin A+\sin B+\sin C)$.

3. Let a circle be given with center O and radius r. Let P be a given point, and let d be the distance OP. Let l be a line through P intersecting the circle at the points A and A'. Show that

   (a) If P is inside the circle, then $PA \times PA' = r^2 - d^2$.

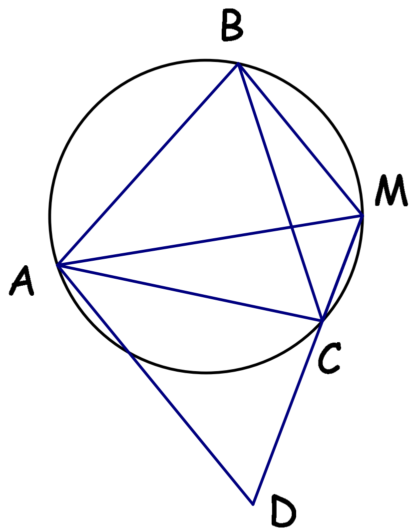
   (b) If P is outside the circle, then $PA \times PA' = d^2 - r^2$.

   Therefore, if we use sensed magnitudes in defining the power of P relative to the circle with radius r, then the power of P relative to this circle is always $d^2 - r^2$.

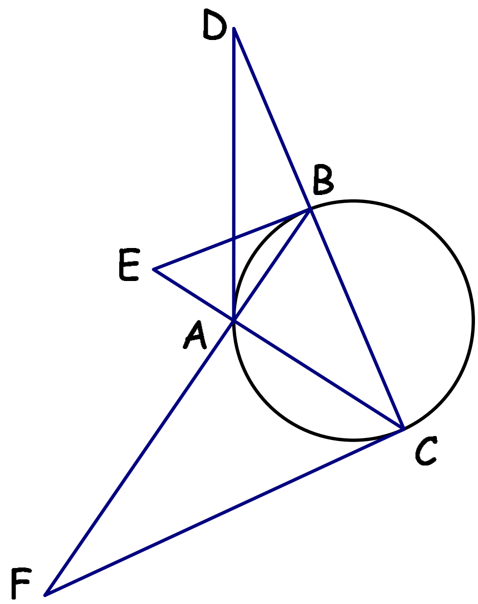
4. Given the circle $\mathcal{C}$ and a real number p, describe the locus of all points P having power p relative to $\mathcal{C}$.

5. Let P be a point and let $\mathcal{C}$ be a circle. Let A and A' be **antipodal** points on the circle (i.e., the line segment $[AA']$ is a diameter of $\mathcal{C}$). Show that the power of P relative to $\mathcal{C}$ is given by the vector dot product $\overrightarrow{PA} \cdot \overrightarrow{PA'}$. (Hint: Note that if O is the center of $\mathcal{C}$, then $\overrightarrow{PA}=\overrightarrow{PO} + \overrightarrow{OA}$ and $\overrightarrow{PA'}=\overrightarrow{PO} - \overrightarrow{OA}$. Apply exercise 3.)

6. Prove Van Schooten's theorem. Namely, let $\triangle ABC$ be an equilateral triangle, and let $\mathcal{C}$ be the circumscribed circle. Let $M \in \mathcal{C}$ be a point on the shorter arc $\widehat{BC}$. Show that $AM = BM + CM$. (Hint: Construct the point D subject to $AM = DM$ and show that $\triangle ABM \cong \triangle ACD$.)



7. The figure to the right shows the triangle $\triangle ABC$ inscribed in a circle. The tangent to the circle at the vertex A meets the line (BC) at D, the tangent to the circle at B meets the line (AC) at E, and the tangent to the circle at C meets the line (AB) at F. Show that D, E, and F are colinear. (Hint: note that $\triangle ACD \sim \triangle BAD$ (why?) and from this you can conclude that $\frac{DB}{DC} = \left(\frac{AB}{AC}\right)^2$. How does this help?)



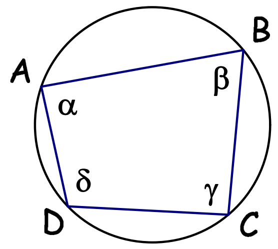
1.3.3 Cyclic quadrilaterals and Ptolemy's theorem

As we have already seen, any triangle can be incribed in a circle; this circle will have center at the circumcenter of the given triangle. It is then natural to ask whether the same can be said for arbitrary polygons. However, a moment's though reveals that this is, in general false even for quadrilaterals. A quadrilateral that can be incribed in a circle is called a **cyclic quadrilateral**.

THEOREM. *The quadrilateral ABCD is cyclic if and only if*

$$A\widehat{B}C + C\widehat{D}A = C\widehat{A}B + B\widehat{C}D = 180^\circ. \quad (1.1)$$

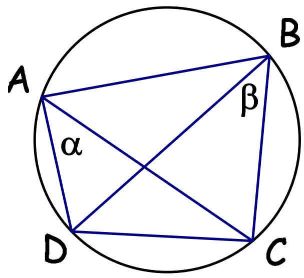
In other words, both pairs of opposite angles add to 180°.



PROOF. If the quadrilateral is cyclic, the result follows easily from the Inscribed Angle theorem. (Draw a picture and check it out!) Conversely, assume that the condition holds true. We let $\mathcal{C}$ be circumscribed circle for the triangle $\triangle ABC$. If D were inside this circle, then clearly we would have $A\widehat{B}C + C\widehat{D}A > 180^\circ$. If D were outside this circle, then $A\widehat{B}C + C\widehat{D}A < 180^\circ$, proving the lemma.

The following is even easier:

THEOREM. *The quadrilateral ABCD is cyclic if and only if* $D\widehat{A}C = D\widehat{B}C$.

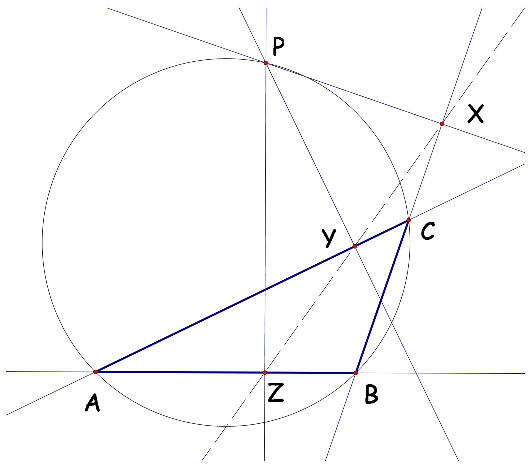


PROOF. The indicated angles open the same arc. The converse is also (relatively) easy.

Simson's line (Wallace's line). There is another line that can be naturally associated with a given triangle $\triangle ABC$, called *Simson's Line* (or sometimes *Wallace's Line*), constructed as follows.

Given the triangle $\triangle ABC$, construct the circumcenter $\mathcal{C}$ and arbitrarily choose a point P on the circle. From P drop perpendiculars to the lines (BC), (AC), and (AB), calling the points of intersection X, Y, and Z, as indicated in the figure below.

THEOREM. *The points X, Y, and Z, constructed as above are colinear. The resulting line is called* **Simson's line** (*or* **Wallace's line**) *of the triangle $\triangle ABC$.*

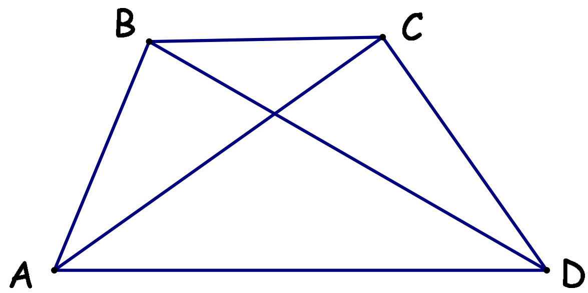


PROOF. Referring to the diagram we note that $P\widehat{Z}B$ and $P\widehat{X}B$ are both right angles. This implies that $X\widehat{P}Z + Z\widehat{B}X = 180^\circ$ and so the quadrilateral $PXBZ$ is cyclic. As a result, we conclude that $P\widehat{X}Z = P\widehat{B}Z$. Likewise, the smaller quadrilateral $PXCY$ is cyclic and so $P\widehat{C}A = P\widehat{C}Y = P\widehat{X}Y$. Therefore,

$$\begin{aligned} P\widehat{X}Z &= P\widehat{B}Z \\ &= P\widehat{B}A \\ &= P\widehat{C}A \quad \text{(angles open the same arc)} \\ &= P\widehat{C}Y \\ &= P\widehat{X}Y, \end{aligned}$$

which clearly implies that X, Y, and Z are coliner.

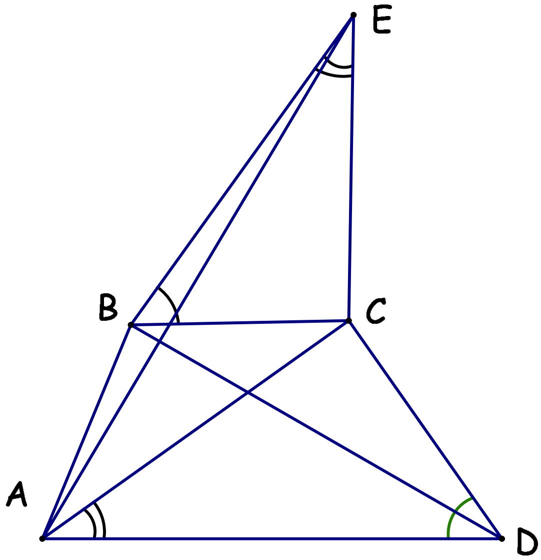
PTOLEMY'S THEOREM.*If the quadrilateral $ABCD$ is cyclic, then the product of the two diagonals is equal to the sum of the products of the opposide side lengths:*



$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

When the quadrilateral is not cyclic, then

$$AC \cdot BD < AB \cdot CD + AD \cdot BC.$$



PROOF. Whether or not the quadrilateral is cyclic, we can construct the point E so that $\triangle CAD$ and $\triangle CEB$ are similar. This immediately implies that

$$\frac{CE}{CA} = \frac{CB}{CD} = \frac{BE}{DA},$$

from which we obtain

$$BE = \frac{CB \cdot DA}{CD}. \qquad (1.2)$$

Also, it is clear that $E\widehat{C}A = B\widehat{C}D$; since also

$$\frac{CD}{CA} = \frac{CB}{CE},$$

we may infer that $\triangle ECA \sim \triangle BCD$. Therefore,

$$\frac{EA}{BD} = \frac{CA}{CD},$$

forcing

$$EA = \frac{CA \cdot DB}{CD}. \qquad (1.3)$$

If it were the case that $ABCD$ were cyclic, then by (1.1) we would have

$$C\widehat{B}E + A\widehat{B}C = C\widehat{D}A + A\widehat{B}C = 180^\circ.$$

But this clearly implies that A, B, and E are colinear, forcing

$$EA = AB + BE$$

Using (1.2) and (1.3) we get

$$\frac{CA \cdot DB}{CD} = AB + \frac{CB \cdot DA}{CD},$$

proving the first part of Ptolemy's theorem.

Assume, conversely, that $ABCD$ is not cyclic, in which case it follows that

$$C\widehat{B}E + A\widehat{B}C \;=\; C\widehat{D}A + A\widehat{B}C \neq 180^\circ.$$

This implies that the points A, B, and E form a triangle from which it follows that $EA < AB + BE$. As above we apply (1.2) and (1.3) and get

$$\frac{CA \cdot DB}{CD} < AB + \frac{CB \cdot DA}{CD},$$

and so

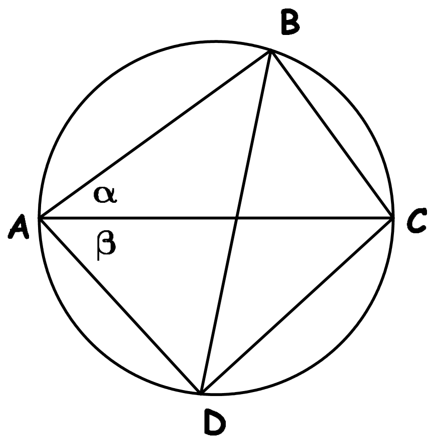
$$CA \cdot DB < AB \cdot CD + CB \cdot DA,$$

proving the converse.

COROLLARY. **(The Addition Formulas for Sine and Cosine)** *We have, for angles α and β, that*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha; \;\; \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

PROOF. We shall draw a cyclic quadrilateral inside a circle having diameter $AC = 1$ (as indicated), and leave the details to the reader. (Note that by Exercise 3 on page 30, we have that $BD = \sin(\alpha + \beta)$ (see the figure). To obtain the addition formula for cos, note that $\cos\alpha = \sin(\alpha + \pi/2)$.)



EXERCISES

1. $[AB]$ and $[AC]$ are chords of a circle with center O. X and Y are the midpoints of $[AB]$ and $[AC]$, respectively. Prove that O, X, A, and Y are concyclic points.

2. Derive the Pythagorean Theorem from Ptolemy's theorem. (This is very easy!)

3. Derive Van Schooten's theorem (see page 35) as a consequence of Ptolemy's theorem. (Also very easy!)

4. Use the addition formula for the sine to prove that if $ABCD$ is a cyclic quadrilateral, then $AC \cdot BD = AB \cdot DC + AD \cdot BC$.

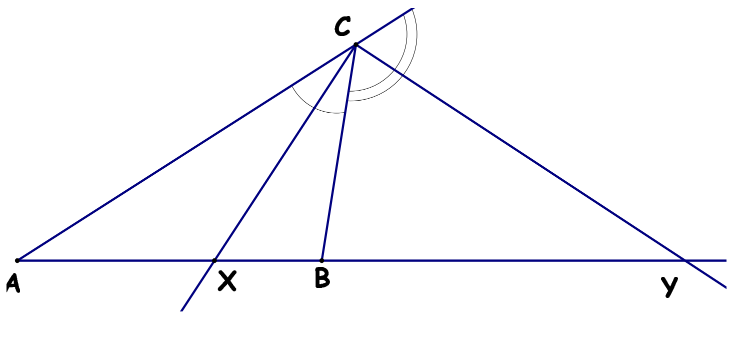
5. Show that if $ABCD$ is a cyclic quadrilateral with side length a, b, c, and d, then the area K is given by

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = (a+b+c+d)/2$ is the semiperimeter.[8]

1.4 Internal and External Divisions; the Harmonic Ratio

The notion of **internal** and **external** division of a line segment $[AB]$ is perhaps best motivated by the familiar picture involving internal and external bisection of a triangle's angle (see the figure to the right). Referring to this figure, we say that the point X divides the segment $[AB]$ **internally** and that the point Y divides the segment $[AB]$ **externally**. In general, if A, B, and X are colinear points, we



[8]This result is due to the ancient Indian mathematician Brahmagupta (598–668).

set $A;X;B = \frac{AX}{XB}$ (signed magnitudes); if $A;X;B > 0$ we call this quantity the **internal division** of $[AB]$, and if $A;X;B < 0$ we call this quantity the **external division** of $[AB]$. Finally, we say that the colinear points A, B, X, and Y are in a **harmonic ratio** if

$$A;X;B \;=\; -A;Y;B;$$

that is to say, when

$$\frac{AX}{XB} \;=\; -\frac{AY}{YB} \quad \text{(signed magnitudes).}$$

It follows immediately from the Angle Bisector Theorem (see page 15) that when (BX) bisects the interior angle at C in the figure above and (BY) bisects the exterior angle at C, then A, B, X, and Y are in harmonic ratio.

Note that in order for the points A, B, X, and Y be in a harmonic ratio it is necessary that one of the points X, Y be interior to $[AB]$ and the other be exterior to $[AB]$. Thus, if X is interior to $[AB]$ and Y is exterior to $[AB]$ we see that A, B, X, and Y are in a harmonic ratio precisely when

$$\text{Internal division of } [AB] \text{ by } X \;=\; -\text{External division of } [AB] \text{ by } Y.$$

Exercises

1. Let A, B, and C be colinear points with $(A;B;C)(B;A;C) = -1$. Show that the **golden ratio** is the positive factor on the left-hand side of the above equation.

2. Let A, B, and C be colinear points and let $\lambda = A;B;C$. Show that under the 6=3! permutations of A, B, C, the possible values of $A;B;C$ are

$$\lambda,\ \frac{1}{\lambda},\ -(1+\lambda),\ -\frac{1}{1+\lambda},\ -\frac{1+\lambda}{\lambda},\ -\frac{\lambda}{1+\lambda}.$$

3. Let A, B, X, and Y be colinear points. Define the **cross ratio** by setting

$$[A, B; X, Y] \;=\; \frac{AX}{AY} \cdot \frac{YB}{XB} \quad \text{(signed magnitudes)}.$$

Show that the colinear points A, B, X, and Y are in harmonic ratio if $[A, B; X, Y] = -1$.

4. Show that for colinear points A, B, X, and Y one has

$$[A, B; X, Y] = [X, Y; A, B] = [B, A; Y, X] = [Y, X; B, A].$$

Conclude from this that under the $4! = 24$ permutations of A, B, X, and Y, there are at most 6 different values of the cross ratio.

5. Let A, B, X, and Y be colinear points, and set $\lambda = [A, B; X, Y]$. Show that under the $4!$ permutations of A, B, X, and Y, the possible values of the cross ratio are

$$\lambda, \ \frac{1}{\lambda}, \ 1 - \lambda, \ \frac{1}{1 - \lambda}, \ \frac{\lambda}{\lambda - 1}, \ \frac{\lambda - 1}{\lambda}.$$

6. If A, B, X, and Y are in a harmonic ratio, how many possible values are there of the cross ratio $[A, B; X, Y]$ under permutations?

7. Let A and B be given points.

   (a) Show that the locus of points $\{M \mid MP = 3MQ\}$ is a circle.

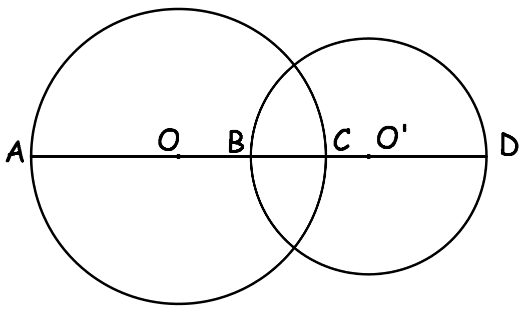
   (b) Let X and Y be the points of intersection of (AB) with the circle described in part (a) above. Show that the points A, B, X, and Y are in a harmonic ratio.

8. Show that if $[A, B; X, Y] = 1$, then either $A = B$ or $X = Y$.

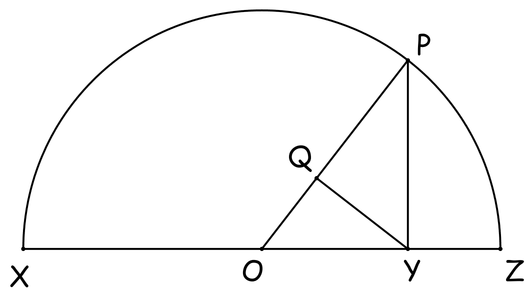
9. The **harmonic mean** of two real numbers is a and b is given by $\dfrac{2ab}{a + b}$. Assume that the points A, B, X, and Y are in a harmonic

ratio. Show that AB is the harmonic mean of AX and AY.[9]

10. The figure to the right depicts two circles having an **orthogonal intersection**. (What should this mean?) Relative to the diagram to the right (O and O' are the centers), show that A, C, B, and D are in a harmonic ratio.



11. The figure to the right shows a semicircle with center O and diameter XZ. The segment $[PY]$ is perpendicular to $[XZ]$ and the segment $[QY]$ is perpendicular to $[OP]$. Show that PQ is the harmonic mean of XY and YZ.



1.5 The Nine-Point Circle

One of the most subtle mysteries of Euclidean geometry is the existence of the so-called "nine-point circle," that is a circle which passes through nine very naturally pre-prescribed points.

To appreciate the miracle which this presents, consider first that arranging for a circle to pass through three noncollinear points is, of course easy: this is the circumscribed circle of the triangle defined by these points (and having center at the circumcenter). That a circle will not, in general pass through four points (even if no three are colinear)

[9] The harmonic mean occurs in elementary algebra and is how one computes the average rate at which a given task is accomplished. For example, if I walk to the store at 5 km/hr and walk home at a faster rate of 10 km/hr, then the average rate of speed which I walk is given by

$$\frac{2\times 5\times 10}{5+10} = \frac{20}{3} \text{ km/hr.}$$

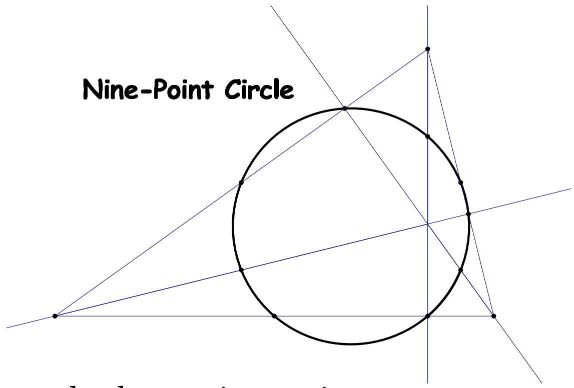
we need only recall that not all quadrilaterals are cyclic. Yet, as we see, if the nine points are carefully—but naturally—defined, then such a circle does exist!

THEOREM. *Given the triangle $\triangle ABC$, construct the following nine points:*

(i) The bases of the three altitudes;

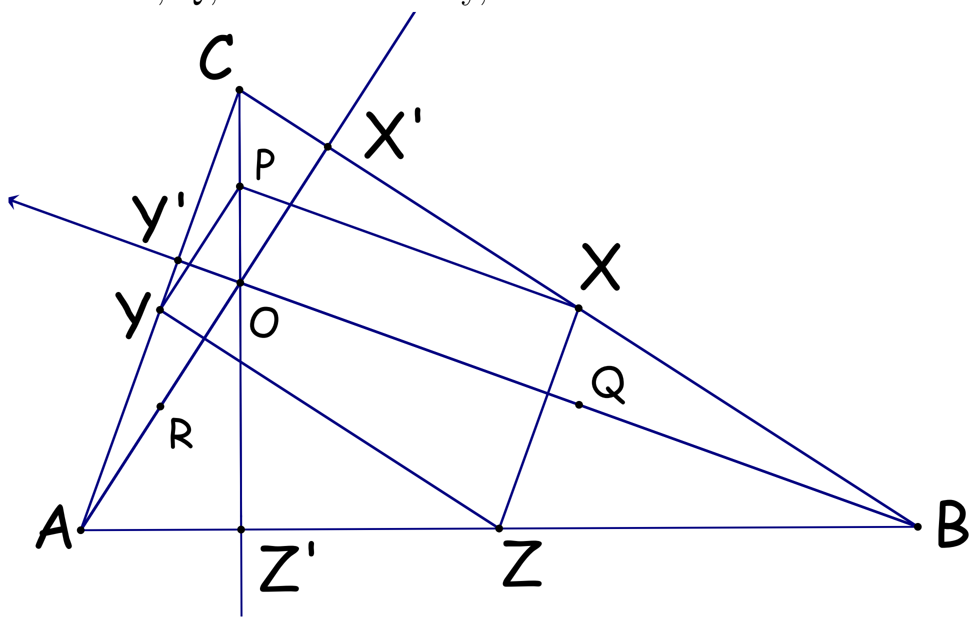
(ii) The midpoints of the three sides;

(iii) The midpoints of the segments joining the orthocenter to each of the vertices.



Then there is a unique circle passing through these nine points.

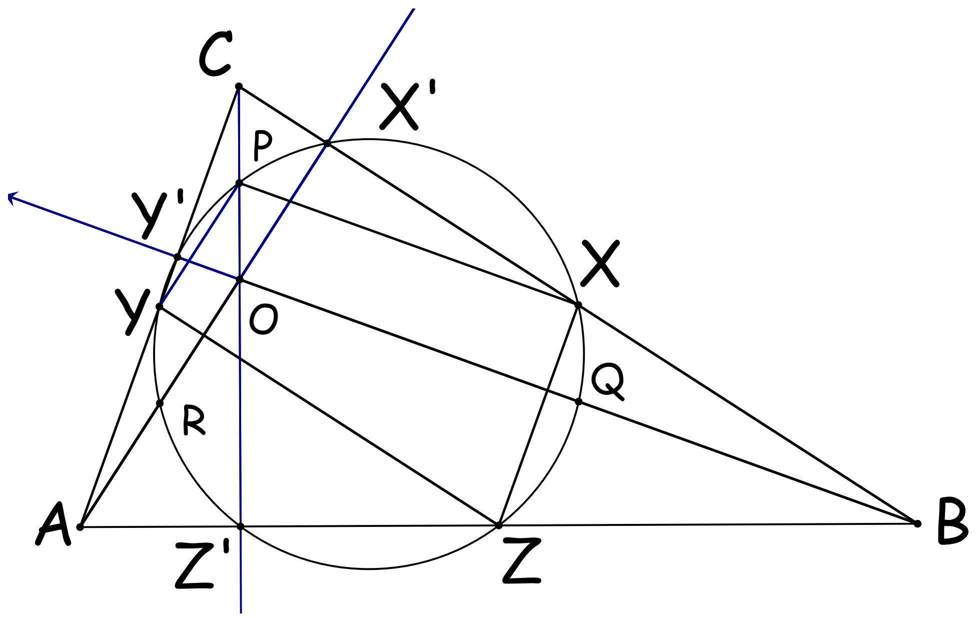
PROOF. Refer to the picture below, where A, B, and C are the vertices, and X, Y, and Z are the midpoints. The midpoints referred to in (iii) above are P, Q, and R. Finally, O is the orthocenter of $\triangle ABC$.



By the Midpoint Theorem (Exercise 3 on page 6 applied to $\triangle ACO$, the line (YP) is parallel to (AX'). Similarly, the line (YZ) is parallel to (BC). This implies immediately that $\angle PYZ$ is a right angle. SImilarly, the Midpoint Theorem applied to $\triangle ABC$ and to $\triangle CBO$ implies that

(XZ) and (AC) are parallel as are (PX) and (BY'). Therefore, $\angle PXZ$ is a right angle. By the theorem on page 35 we conclude that the quadrilateral $YPXZ$ is cyclic and hence the corresponding points all lie on a common circle. Likewise, the quadrilateral $PXZZ'$ is cyclic forcing its vertices to lie on a common circle. As three non-collinear points determine a unique circle (namely the circumscribed circle of the corresponding triangle—see Exercise 8 on page 17) we have already that P, X, Y, Z, and Z' all lie on a common circle.

In an entirely analogous fashion we can show that the quadrilaterals $YXQZ$ and $YXZR$ are cyclic and so we now have that P, Q, R, X, Y, Z, and Z' all lie on a common circle. Further analysis of cyclic quadrilaterals puts Y' and Z' on this circle, and we're done!



Note, finally, that the nine-point circle of $\triangle ABC$ lies on this triangle's Euler line (see page 22).

EXERCISES.

1. Prove that the center of the nine-point circle is the circumcenter of $\triangle XYZ$.

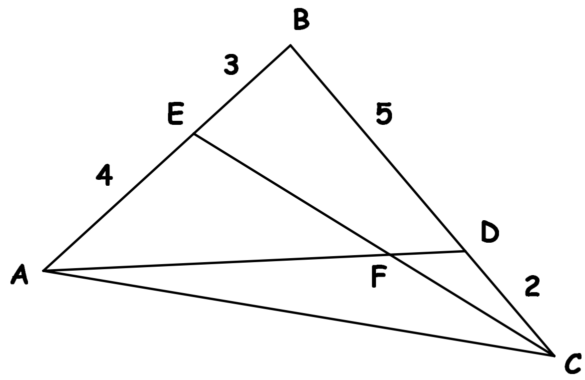
2. Referring to the above diagram, prove that the center of the nine-point circle lies at the midpoint of the segment $[NO]$, where N is the orthocenter of $\triangle ABC$.

3. Given $\triangle ABC$, let O be its orthocenter. Let $\mathcal{C}$ be the nine-point circle of $\triangle ABC$, and let $\mathcal{C}'$ be the circumcenter of $\triangle ABC$. Show that $\mathcal{C}$ bisects any line segment drawn from O to $\mathcal{C}'$.

1.6 Mass point geometry

Mass point geometry is a powerful and useful viewpoint particularly well suited to proving results about ratios—especially of line segments. This is often the province of the Ceva and Menelaus theorems, but, as we'll see, the present approach is both easier and more intuitive.

Before getting to the definitions, the following problem might help us fix our ideas. Namely, consider $\triangle ABC$ with Cevians $[AD]$ and $[CE]$ as indicated to the right. Assume that we have ratios $BE : EA = 3 : 4$ and $CD : DB = 2 : 5$. Compute the ratios $EF : FC$ and $DF : FA$.



Both of the above ratios can be computed fairly easily using the converse to Menalaus' theorem. First consider $\triangle CBE$. From the converse to Menelaus' theorem, we have, since A, F, and D are colinear, that (ignoring the minus sign)

$$1 = \frac{2}{5} \times \frac{7}{4} \times \frac{EF}{FC},$$

forcing $EF : FC = 10 : 7$.

Next consider $\triangle ABD$. Since the points E, F, and C are colinear, we have that (again ignoring the minus sign)

$$1 = \frac{4}{3} \times \frac{7}{2} \times \frac{DF}{FA},$$

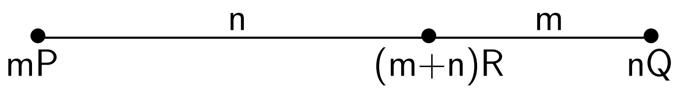
and so $DF : FA = 3 : 14$.

Intuitively, what's going on can be viewed in the following very tangible (i.e., physical) way. Namely, if we assign "masses" to the points of $\triangle ABC$, say

$$A \text{ has mass } \frac{3}{2}; \; B \text{ has mass } 2; \quad \text{and } C \text{ has mass } 5,$$

then the point E is at the center of mass of the weighted line segment $[AB]$ and has mass $\dfrac{7}{2}$, and D is at the center of mass of the weighted line segment $[BC]$ and has mass 7. This suggests that F should be at the center of mass of both of the weighted line segments $[CE]$ and $[AD]$, and should have total mass $\frac{17}{2}$. This shows why $DF : FA = \frac{3}{2} : 7 = 3 : 14$ and why $EF : FC = 5 : \frac{7}{2} = 10 : 7$.

We now formalize the above intuition as follows. By a **mass point** we mean a pair (n, P)—usually written simply as nP—where n is a positive number and where P is a point in the plane.[10] We define an **addition** by the rule: $mP + nQ = (m + n)R$, where the point R is on the line segment $[PQ]$, and is at the center of mass inasmuch as $PR : RQ = n : m$. We view this as below.

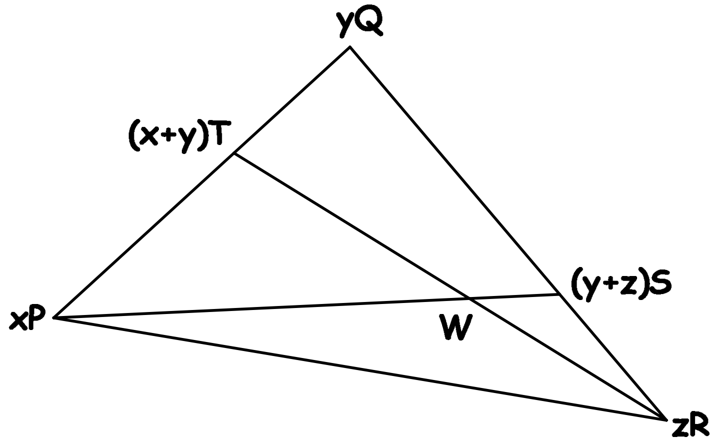


It is clear that the above addition is **commutative** in the sense that $xP + yQ = yQ + xP$. However, what isn't immediately obvious is that this addition is **associative**, i.e., that $xP+(yQ+zR) = (xP+yQ)+zR$ for positive numbers x, y, and z, and points P, Q, and R. The proof is easy, but it is precisely where the converse to Menelaus' theorem comes in! Thus, let

$$yQ + zR = (y + z)S, \quad xP + yQ = (x + y)T.$$

Let W be the point of intersection of the Cevians $[PS]$ and $[RT]$.

[10] Actually, we can take P to be in higher-dimensional space, if desired!



Applying the converse to Menelaus' theorem to the triangle $\triangle PQS$, we have, since T, W, and R are colinear, that (ignore the minus sign)

$$1 = \frac{PT}{TQ} \times \frac{QR}{RS} \times \frac{SW}{WP} = \frac{y}{x} \times \frac{y+z}{y} \times \frac{SW}{WP}.$$

This implies that $PW : WS = (y+z) : x$, which implies that

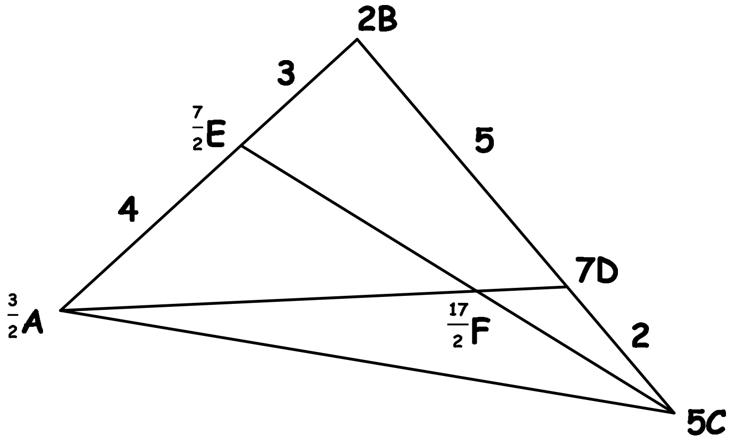
$$(x+y+z)W = xP + (y+z)S = xP + (yQ + zR).$$

Similarly, by applying the converse of Menelaus to $\triangle QRT$, we have that $(x+y+z)W = (x+y)T + zR = (xP + yQ) + zR$, and we're done, since we have proved that

$$xP + (yQ + ZR) = (x+y+z)W = (xP + yQ) + zR.$$

The point of all this is that given mass points xP, yQ, and zR, we may unambiguously denote the "center of mass" of these points by writing $xP + yQ + zR$.

Let's return one more time to the example introduced at the beginning of this section. The figure below depicts the relevant information. Notice that the assigments of masses to A, B, and C are uniquely determined up to a nonzero multiple.



The point F is located at the center of mass—in particular it is on the line segments $[AD]$ and $[CE]$; furthermore its total mass is $\frac{17}{2}$. As a result, we have that $AF : FD = 7 : \frac{3}{2} = 14 : 3$ and $CF : FE = \frac{7}{2} : 5 = 14 : 10$, in agreement with what was proved above.

We mention in passing that mass point geometry can be used to prove Ceva's theorem (and its converse) applied to $\triangle ABC$ when the Cevians $[AX]$, $[BY]$, and $[CZ]$ meet the triangle's sides $[BC]$, $[AC]$, and $[AB]$, respectively. If we are given that

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1,$$

we assign mass ZB to vertex A, mass AZ to vertex B, and mass $\frac{AZ \cdot BA}{XC}$ to vertex C. Since $ZB : \frac{AZ \cdot BX}{XC} = \frac{CY}{YA}$, we see that the center of mass will lie on the intersection of the three Cevians above. Conversely, if we're given the three concurrent Cevians $[AX]$, $[BY]$, and $[CZ]$, then assigning masses as above will place the center of mass at the intersection of the Cevians $[AX]$ and $[CZ]$. Since the center of mass is also on the Cevian $[BY]$, we infer that

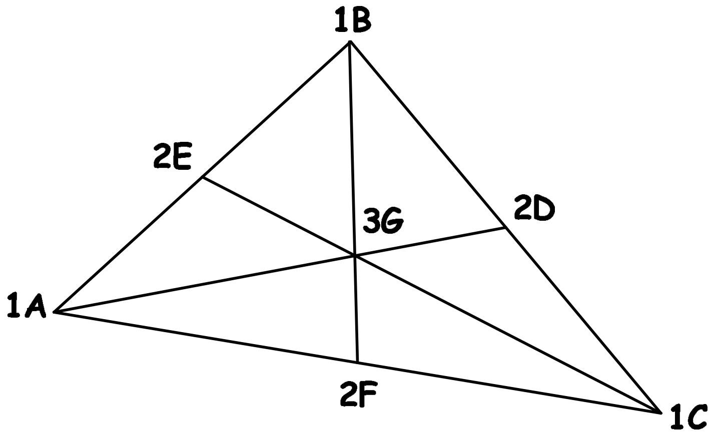
$$\frac{CY}{YA} = \frac{ZB \cdot XC}{AZ \cdot BX},$$

and we're done!

We turn to a few examples, with the hopes of conveying the utility of this new approach. We emphasize: the problems that follow can all be solved without mass point geometry; however, the mass point approach is often simpler and more intuitive!

EXAMPLE 1. *Show that the medians of $\triangle ABC$ are concurrent and the point of concurrency (the centroid) divides each median in a ratio of 2:1.*

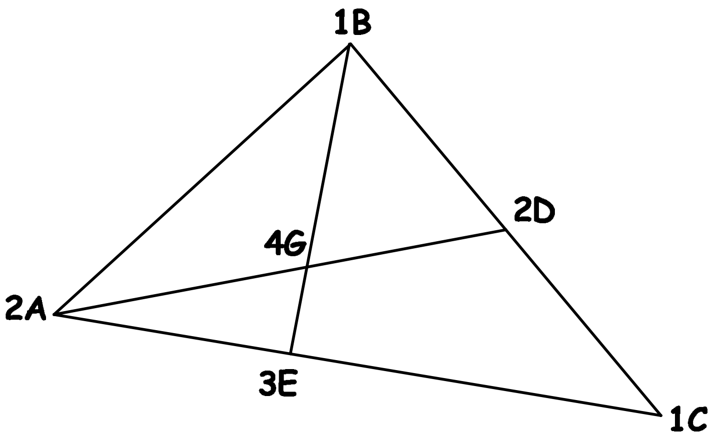
SOLUTION. We assign mass 1 to each of the points A, B, and C, giving rise to the following weighted triangle:



The point G, begin the center of mass, is on the intersection of all three medians—hence they are concurrent. The second statement is equally obvious as $AG : GD = 2 : 1$; similarly for the other ratios.

EXAMPLE 2. *In $\triangle ABC$, D is the midpoint of $[BC]$ and E is on $[AC]$ with $AE : EC = 1 : 2$. Letting G be the intersections of the Cevians $[AD]$ and $[BE]$, find $AG : GD$ and $BG : GE$.*

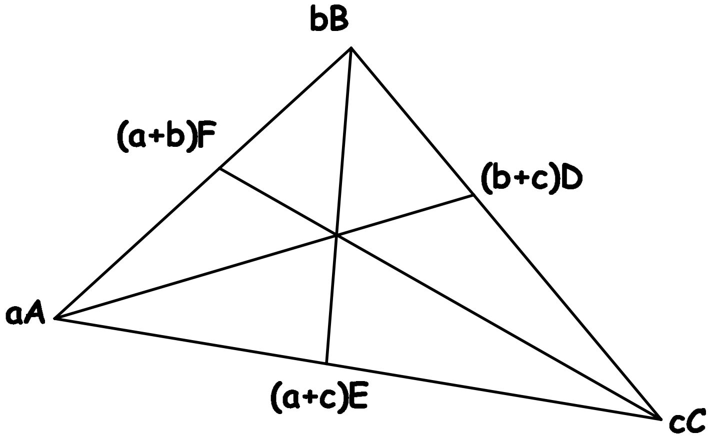
SOLUTION. The picture below tells the story:



From the above, one has $AG : GD = 1 : 1$, and $BG : GE = 3 : 1$.

Example 3. *Prove that the angle bisectors of $\triangle ABC$ are concurrent.*

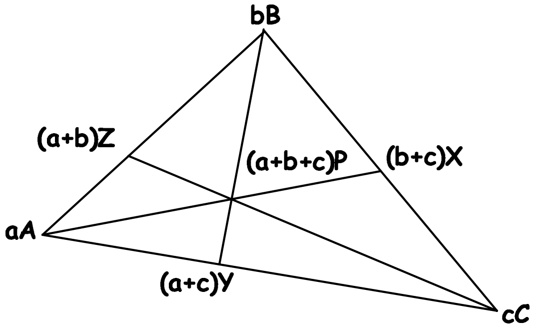
Proof. Assume that $AB = c$, $AC = b$, $BC = a$ and assign masses a, b, and c to points A, B, and C, respectively. We have the following picture:

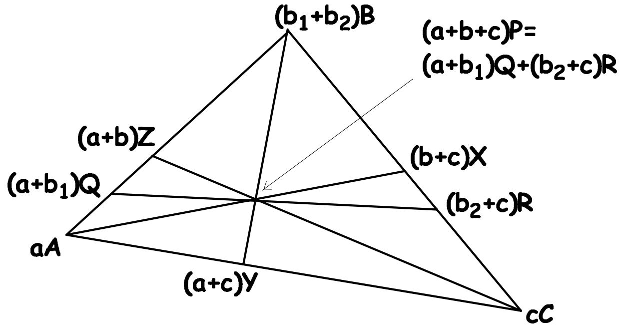


Note that as a result of the Angle Bisector Theorem (see page 15) each of the Cevians above are angle bisectors. Since the center of mass is on each of these Cevians, the result follows.

The above applications have to do with Cevians. The method of mass point geometry also can be made to apply to **transversals**, i.e., lines through a triangle not passing through any of the vertices. We shall discuss the necessary modification (i.e., **mass splting**) in the context of the following example.

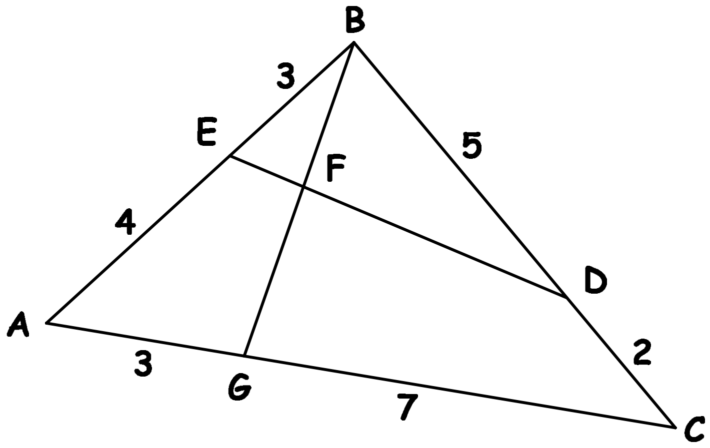
Solution. The above examples were primarily concerned with computing ratios along particular Cevians. In case a **transversal** is involved, then the method of "mass splitting" becomes useful. To best appreciate this, recall that if in the triangle $\triangle ABC$ we assign mass a to A, b to B, and c to C, then the center of mass P is located on the intersection of the three Cevians (as depicted below). However, suppose that we "split" the mass b at B into two components $b = b_1 + b_2$, then the center of mass P will not only lie at the intersection of the concurrent Cevians, it will also lie on the transversal $[XZ]$; see below:



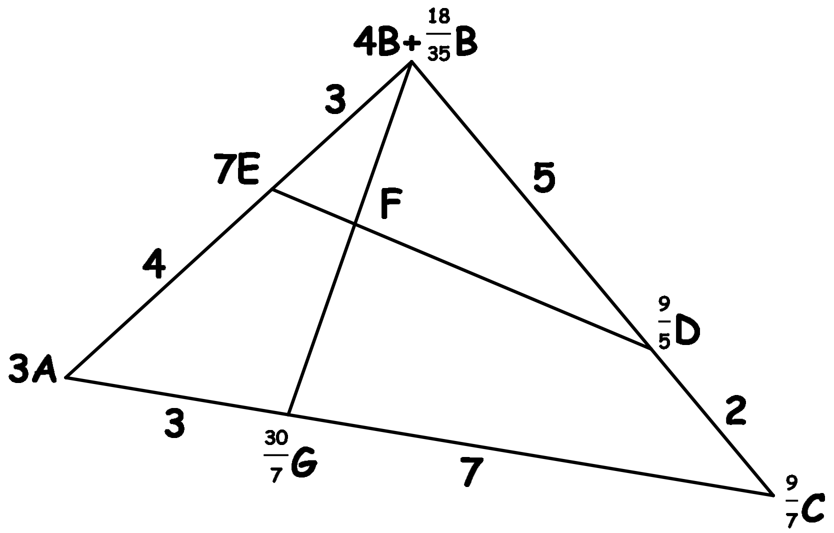


Note that in the above diagram, $QP : PR = (b_2 + c) : (a + b_1)$ because P is the center of mass of $[QR]$.

Example 4. *In the figure below, compute $EF : FD$ and $BF : FG$.*



Solution. We shall arrange the masses so that the point F is the center of mass. So we start by assigning weights to A and B to obtain a balance $[AB]$ at E: clearly, assigning mass 4 to B and 3 to A will accomplish this. Next, to balance $[AC]$ at G we need to assign mass $\frac{9}{7}$ to C. Finally, to balance $[BC]$ at D, we need another mass of $\frac{18}{35}$ at B, producing a total mass of $4 + \frac{18}{35}$ at B. The point F is now at the center of mass of the system! See the figure below:



From the above, it's easy to compute the desired ratios:

$$EF : FD = \frac{9}{5} : 7 = 9 : 35 \quad \text{and} \qquad BF : FG = \frac{30}{7} : \frac{158}{35} = 75 : 79.$$

Exercises

1. In $\triangle ABC$, D is the midpoint of $[BC]$ and E is on $[AC]$ with $AE : EC = 1 : 2$. Let G be the intersection of segments $[BE]$ and $[AD]$ and find $AG : GD$ and $BG : GE$.

2. In $\triangle ABC$, D is on $[AB]$ with $AD = 3$ and $DB = 2$. E is on $[BC]$ iwht $BE = 3$ and $EC = 4$. Compute $EF : FA$.

3. In quadrilateral $ABCD$, E, F, G, and H are the trisection points of $[AB]$, $[BC]$, $[CD]$, and DA nearer A, C, C, and A, respectively. Show that $EFGH$ is a parallogram. (Show that the diagonals bisect each other.)

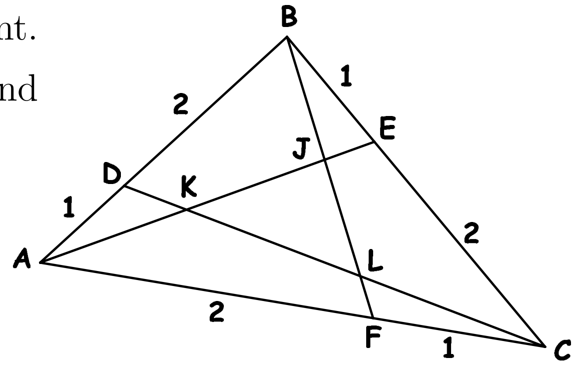
4. Let $[AD]$ be an altitude in $\triangle ABC$, and assume that $\angle B = 45°$ and $\angle C = 60°$. Assume that F is on $[AC]$ such that $[BF]$ bisects $\angle B$. Let E be the intersection of $[AD]$ and $[BF]$ and compute $AE : ED$ and $BE : EF$.

5. [11] In triangle ABC, point D is on $[BC]$ with $CD = 2$ and $DB = 5$, point E is on $[AC]$ with $CE = 1$ and $EA = 3, AB = 8$, and $[AD]$ and $[BE]$ intersect at P. Points Q and R lie on $[AB]$ so that $[PQ]$ is parallel to $[CA]$ and $[PR]$ is parallel to $[CB]$. Find the ratio of the area of $\triangle PQR$ to the area of $\triangle ABC$.

6. In $\triangle ABC$, let E be on $[AC]$ with $AE : EC = 1 : 2$, let F be on $[BC]$ with $BF : FC = 2 : 1$, and let G be on $[EF]$ with $EG : GF = 1 : 2$. Finally, assume that D is on $[AB]$ with C, D, G colinear. Find $CG : GD$ and $AD : DB$.

7. In $\triangle ABC$, let E be on $[AB]$ such that $AE : EB = 1 : 3$, let D be on $[BC]$ such that $BD : DC = 2 : 5$, and let F be on $[ED]$ such that $EF : FD = 3 : 4$. Finally, let G be on $[AC]$ such that the segment $[BG]$ passes through F. Find $AG : GC$ and $BF : FG$.

8. You are given the figure to the right.



   (a) Show that $BJ : JF = 3 : 4$ and $AJ : JE = 6 : 1$.

   (b) Show that $DK : KL : LC = EJ : JK : KA = FL : LJ : JB = 1 : 3 : 3$.

   (c) Show that the area of $\triangle JKL$ is one-seventh the area of $\triangle ABC$.

   (Hint: start by assigning masses 1 to A, 4 to B and 2 to C.)

9. Generalize the above result by replacing "2" by n. Namely, show that the area ratio

$$\text{area}\,\triangle JKL : \text{area}\,\triangle ABC = (n-1)^3 : (n^3 - 1).$$

   (This is a special case of **Routh's theorem** .)

[11] This is essentially problem #13 on the 2002 American Invitational Mathematics Examination (II).