1 Angles in Geometry

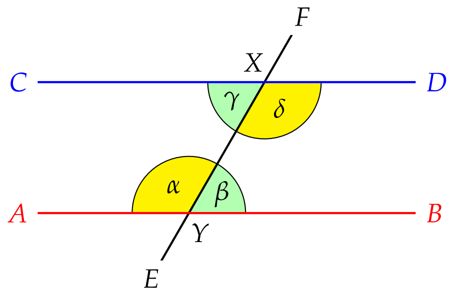
Geometry is one of the most famous parts of mathematics and often the least understood. In this section, you will get better at angles, from simple angle theorems, but also through similar and congruent triangles. Here are a few tips for you when you start doing geometry:

- Draw BIG diagrams. As big as your hand. Small diagrams will hurt your eyes, won't be accurate and you will not be able to see anything interesting in them.
- You don't need to use a ruler or compass. Practice so that your free hand looks quite accurate.
- NEVER assume something that is not in the question. If the question does not say something is equal, but it looks like it, then DO NOT assume it is true. Instead, prove it is true!
- Try not to draw things so that they look isosceles or equilateral. Avoid having angles look like 90° or sidelengths looking equal. Draw scalene triangles!
- Draw BIG diagrams. Very important.

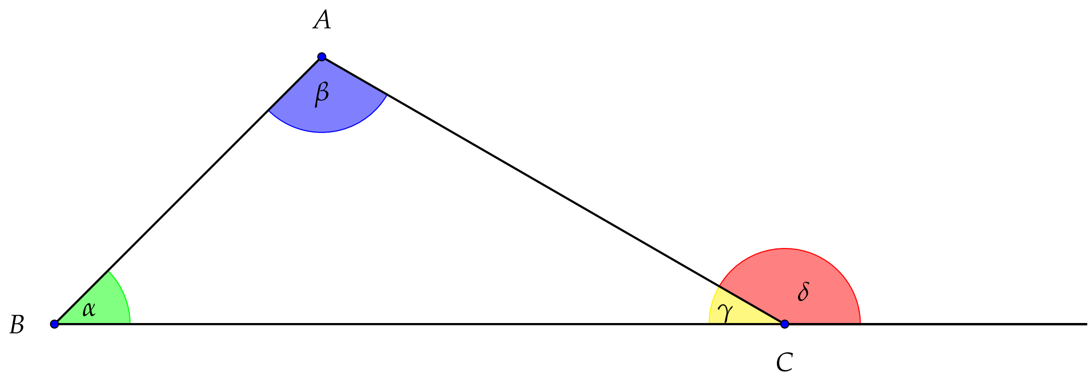
1.1 Angle Theorems

For the following section, look at the diagram below. Assume lines AB and CD are parallel.

- Alternate angles $\beta = \gamma$ or $\alpha = \delta$.
- Cointerior angles $\alpha + \gamma = 180^\circ = \delta + \beta$.
- Corresponding angles $\alpha = \angle CXF$ or $\beta = \angle FXD$.
- Supplementary angles $\alpha + \beta = 180^\circ = \gamma + \delta$.
- Vertically opposite angles $\gamma = \angle FXD$ or $\alpha = \angle EYB$.



Note the following diagram as well for the points below.



- Angle sum in a triangle is $\alpha + \beta + \gamma = 180^\circ$.
- Exterior angle theorem $\alpha + \beta = \delta$.
- Angle sum in n-gon $= (n-2) \times 180^\circ$.

You need to be familiar with standard types of shapes.

- Types of triangles: isosceles, equilateral, right, acute, obtuse and scalene.
- Types of quadrilaterals: square, rectangle, rhombus, parallelogram, trapezium, kite and cyclic.

1.2 Congruent Triangles

Definition. Two triangles ABC and DEF are congruent if at least one of the following cases holds:

- (SSS) All corresponding sides are equal. $AB = DE$, $AC = DF$ and $BC = EF$.
- (SAS) Two pairs of equal sides with the angles between them equal. For example, we could have $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$.
- (AAS) Two equal angles and a corresponding pair of equal sides. For example, $AB = DE$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$.
- (RHS) Both triangles must be right angled with equal hypotenuses and one other equal pair of sides.
- (SSA+ info) If two corresponding sides are equal and a corresponding pair of angles is equal, but this angle is not between the equal sides, then this is not enough to conclude congruence, but other bits of information can help, such as knowing that the triangles are both obtuse.

We write $\triangle ABC \equiv \triangle DEF$.

Theorem 1.1. *If $\triangle ABC \equiv \triangle DEF$, then all corresponding sides and angles are equal. In particular*

$$\begin{aligned} AB &= DE \qquad & \angle ABC &= \angle DEF \\ AC &= DF \qquad & \angle ACB &= \angle DFE \\ BC &= EF \qquad & \angle BAC &= \angle EDF \end{aligned}$$

Congruent triangles are literally the same shape and size!

1.3 Similar Triangles

Definition. Two triangles ABC and DEF are similar if at least one of the cases from the congruence cases holds after we replace all the conditions involving equal sides with conditions necessitating equal proportions of sides. In particular, we get PPP, PAP, AA and RP_HP. We write $\triangle ABC \sim \triangle DEF$.

Theorem 1.2. *If $\triangle ABC \sim \triangle DEF$ and the sidelengths of $\triangle ABC$ are p times as long as those of $\triangle DEF$, then all corresponding angles are equal and corresponding pairs of sides are in equal proportion. Moreover,* Area_{ABC} *is p^2 as big as* Area_{DEF}*. Written another way,*

$$\frac{\text{Area}_{ABC}}{\text{Area}_{DEF}} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{AC}{DF}\right)^2 = \left(\frac{BC}{EF}\right)^2$$

So, similar triangles are the same shape, but not necessarily the same size!

Problems

1.4 Easy

1. Using the theorem about supplementary angles, prove that the angle sum in a triangle is 180°.
2. Work out what all the conditions for similarity are. [Hint: They are the same as congruence, but every S (equal sides) is replaced by a P (sides in proportion). You should get PPP, PAP, AA, $\mathrm{RH}_P\mathrm{P}$, PPA+info].
3. In rectangle $ABCD$, CB is extended to E such that $CE = CA$. F is the midpoint of AE. Prove $DF \perp FB$.
4. Prove that two equal chords in the same circle must be equidistant from the centre.
5. Let P be a point on circle with diameter AB. Drop the perpendicular from P to meet AB at N. Prove $AP^2 = AN \cdot AB$.
6. Circles Γ_1, Γ_2 intersect at A and B. Let C and D lie on Γ_1. Let lines CB and DB intersect Γ_2 again at E and F. Prove $\triangle ACD \sim \triangle AEF$.

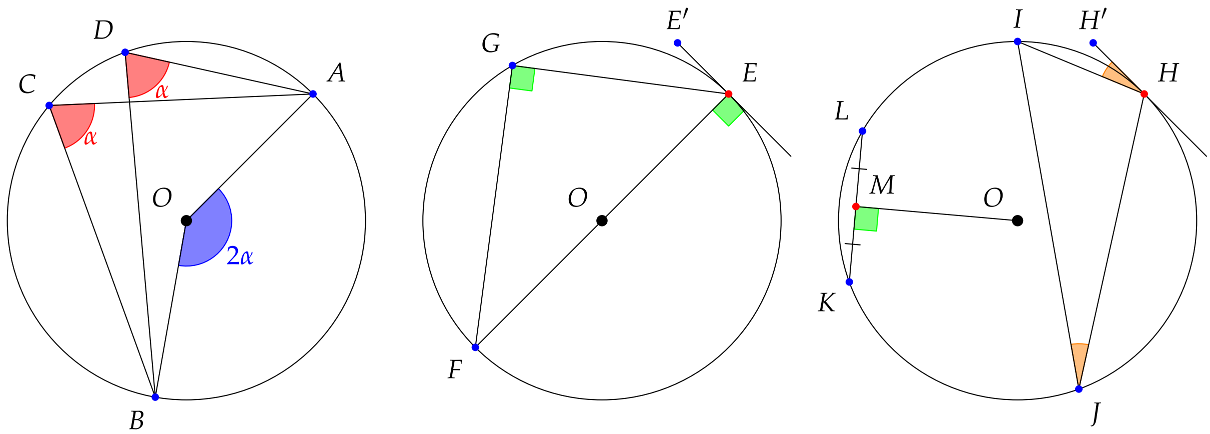
1.5 Hard

1. Let H be the orthocentre (intersection of the altitudes) of $\triangle ABC$ and D be the second intersection of ray AH with circumcircle (circle that goes around triangle touching all the vertices) of ABC. Prove BC bisects HD.
2. Let $\triangle ABC$ be right angled at C. Construct squares on the exteriors of AC and BC, being $ACFG$ and $CBED$. Let AE meet BC at H and BG meet AC at K. Prove $\angle KHC = 45^\circ$.
3. Let ABC be a triangle and let the midpoints of AB, AC be E, F. Let X, Y be points on BC such that $EX \parallel FY$. Prove that $\text{Area}(EFYX) = \frac{1}{2}\,\text{Area}\,ABC$.
4. In $\triangle ABC$, $AB = BC$ and $\angle ACB = 50^\circ$. D is a point on AC such that $AD = BD$. E is a point on BD such that $BE = CD$. Find the size of $\angle EAD$.
5. In trapezium $ABCD$, H, K are be points on AC, BD such that $HK \parallel AB \parallel CD$. If $AB = 4, CD = 9, BK = 2, KD = 3$. Find the length of HK.

2 Circle Geometry

2.1 Angles in Circles

There are a few circle angle theorems that we must be familiar with. See the diagram below for reference.



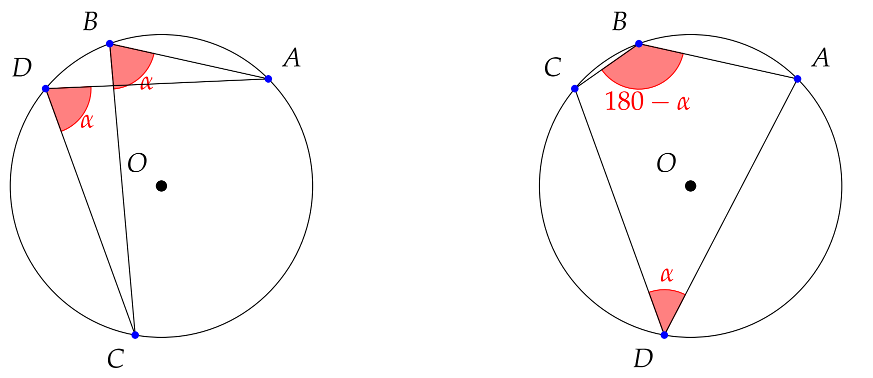
- Equal arcs subtend equal angles: $\angle ACB = \angle ADB = \alpha$. (Also called 'bow-tie theorem')
- Angle at centre is twice angle at circumference: $\angle AOB = 2 \times \angle ACB = 2\alpha$.
- Angle subtended by diameter is $90°$: $\angle FGE = 90°$.
- Line segment from circle centre to point of tangency is perpendicular to tangent: $\angle E'EO = 90°$.
- Line segment from circle centre to midpoint of chord is perpendicular to chord: $\angle KMO = 90°$.
- Alternate segment theorem: $\angle H'HI = \angle HJI$.

2.2 Cyclic Quadrilaterals

It is true that any three non-collinear points are always cyclic. In other words, a circle can always be drawn through these three points. Importantly, there is only one such circle that does this. If we were to add a fourth point and have all four points cyclic, then clearly this fourth point must lie on the circle that passes through the other three points.
So, not every four non-collinear points are cyclic! This is a theorem about the cyclicity of four points:

Theorem 2.1. *Given four points A, B, C and D, they are cyclic iff either*

- $\angle ABC = \angle ADC$ *or*
- $\angle ABC + \angle ADC = 180°$.

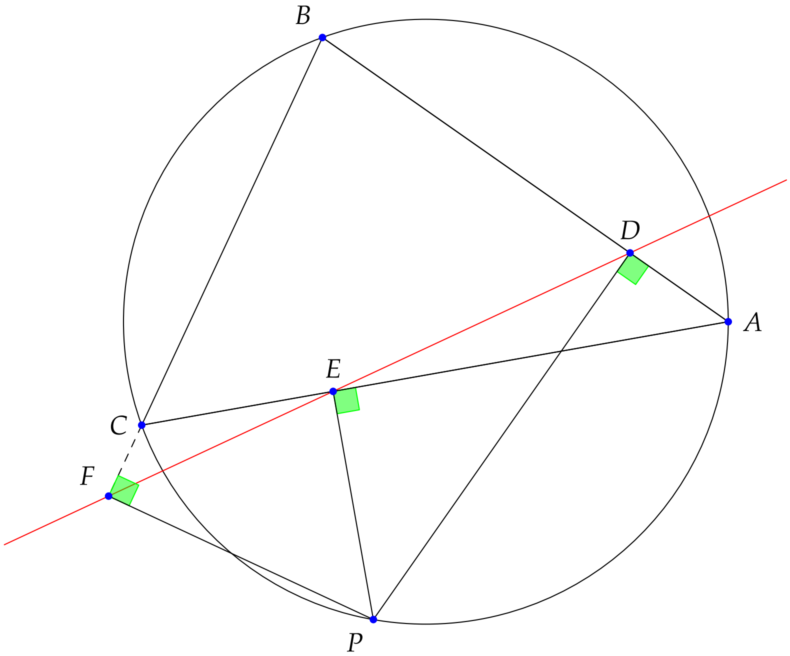


Alternatively, one can think of this as follows: if the four points satisfy the bow-tie theorem, they are cyclic. Alternatively, if we draw in edges and turn them into a quadrilateral, then opposite angles must add up to $180°$ *for it to be cyclic.*

Four points that are cyclic are usually considered together as a cyclic quadrilateral (once you draw in the edges), rather than as four separate points that are cyclic together.

2.3 Simson Line

Theorem 2.2. (Simson Line) *Let* ABC *be a triangle inscribed in a circle. Let* P *be any point on the circle. If* D, E, F *are the feet of the perpendiculars from* P *to* AB, AC, BC *(perhaps extended) respectively, then* D, E, F *are collinear and* DEF *is called the Simson Line of* P *with respect to* ABC*.*



Problems

2.4 Easy

1. Prove the ice cream cone theorem: "Given a circle Γ and a point P outside the circle, there are two points of tangency from P to Γ, call them A and B. The distances PA and PB are equal."

2. Prove that two equal chords in the same circle must be equidistant from the centre.

3. Let ABC be a triangle with circumcircle Γ. Let D, E be points on AB, AC such that DE is parallel to the tangent to Γ at A. Prove that $BDEC$ is cyclic.

4. Let P be a point on circle with diameter AB. Drop the perpendicular from P to meet AB at N. Prove $AP^2 = AN \cdot AB$.

5. Circles Γ_1, Γ_2 intersect at A and B. Let C and D lie on Γ_1. Let lines CB and DB intersect Γ_2 again at E and F. Prove $\triangle ACD \sim \triangle AEF$.

6. Two circles Γ_1 and Γ_2 intersect at points P and Q. A line through P intersects Γ_1 and Γ_2 at points A, B. The line passing through Q and the midpoint of AB, intersects Γ_1 and Γ_2 at points C and D. Prove that $ABCD$ is a parallelogram.

7. Let A, B, C, D lie on circle Γ. Suppose that the tangents to Γ at A, C and line BD are concurrent. Prove $AB \cdot CD = AD \cdot BC$.

8. Let AB be the diameter of circle Γ. Let C and D be points on the tangent of Γ at point B such that B is between C and D. Let AC and AD intersect Γ again at P and Q. Prove $CPQD$ is cyclic.

2.5 Hard

1. Let the incircle of triangle ABC touch the sides BC, AC, AB at P, Q, R respectively. Express the lengths AQ, AR, BR, BP, CP, CQ in terms of the side lengths a, b, c of the triangle. Do the same for an excircle.

2. Prove that the tangents to the circumcircle at the three vertices of a triangle form a triangle similar to the orthic triangle.

3. Triangle ABC is acute. The circle with diameter AB intersects altitude CC' and its extension at points M and N and the circle with diameter AC intersects the altitude BB' at P and Q. Show that $MNPQ$ is cyclic.

4. Let $ABCD$ be a cyclic quadrilateral with the property that its diagonals AC and BD intersect, at M say, in a right angle. Let N be the midpoint of AB and P be the points on CD such that NP and CD are perpendicular. Prove that the points M, N and P are collinear.

5. Let $ABCDEF$ be a cyclic hexagon such that $AB = BC$, $CD = DE$ and $EF = FA$. Prove that lines AD, BE and CF are concurrent.

6. Circles Γ_1, Γ_2 intersect at A and B. Let a common tangent to Γ_1, Γ_2 touch the circles at C and D. Prove the ray AB, when extended, bisects CD.

7. $ABCD$ is a square with an inscribed circle Γ. Γ touches AD at P and BC at Q. PB intersects Γ again at R. M is the midpoint of PR. Prove that $\angle PMQ = 135°$.

3 Lengths and Areas

A lot of the work we have done so far uses angles. Whether it is to show that triangles are similar, or that quadrilaterals are cyclic, I cannot stress enough how important working with angles is. Most diagrams need a very strong angle analysis to really understand what is going on. On the other hand, it is often not enough. Sometimes it is the lengths and areas in a diagram that will make the difference. We have already learnt many theorems that give us results about lengths like similar triangles, congruent triangles, angle bisector theorem, ceva's theorem and Pythagoras. But, as you may well expect, there are more! Use them all!

3.1 Area of a triangle

There are many ways of calculating the area of a triangle. Assume our triangle is $\triangle ABC$ with sidelengths $a(BC), b(AC)$ and $c(AB)$.

Theorem 3.1. *The area of $\triangle ABC$ is given by half the product of one of the sides and the altitude to that side. Thus, if we choose a to be our base and h_a is the altitude from A to BC (possibly extended), then*

$$Area(\triangle ABC) = \frac{ah_a}{2}.$$

If we define the semiperimeter, s, to be half of the perimeter ($s = \frac{1}{2}(a + b + c)$), then we can find the area another way:

Theorem 3.2 (Herron's Formula)**.** *The area of $\triangle ABC$ can also be calculated by a complicated expression discovered in antiquity by Greek mathematician Herron. It is given by*

$$Area(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

That's not all! If the inradius is denoted r, we also have:

Theorem 3.3 (Inradius Area Formula)**.** *The area of $\triangle ABC$ is given by the product of the inradius and the semiperimeter. Thus,*

$$Area(\triangle ABC) = rs.$$

For those of you who have come across trigonometry, there is another formula that I include for completeness.

Theorem 3.4. *The area of $\triangle ABC$ is given by half the product of two sides and the sine of the angle between them. So, choosing the two sides to be a and b yields*

$$Area(\triangle ABC) = \frac{ab\sin(C)}{2}.$$

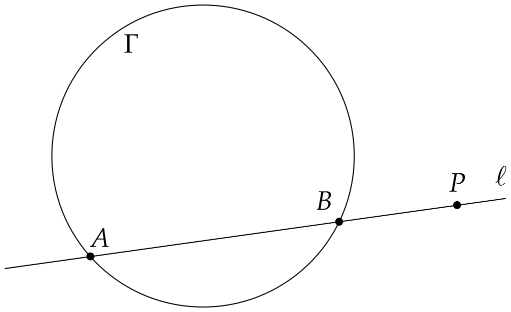
3.2 Power of a Point

Definition. The *power of a point P* with respect to a circle Γ is defined as follows:
draw a line ℓ through P that also intersects Γ in points A and B. Then the power of P with respect to Γ is the value

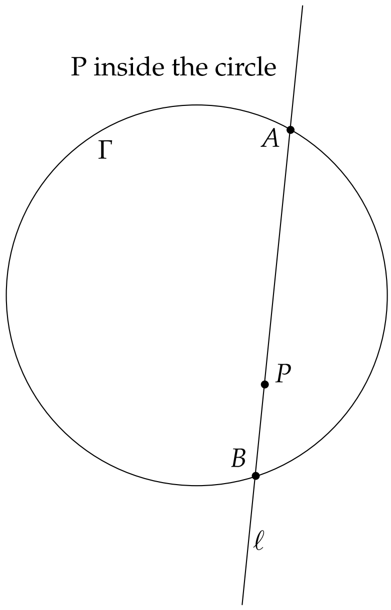
$$PA \times PB.$$

This is true if the point is inside or outside the circle!

P outside the circle



P inside the circle



As you can see from the diagram, there are two diagrams for power of a point, corresponding to P being inside or outside the circle. Your next question should be: but there are lots of lines ℓ that pass through P that intersect the circle Γ! Surely the value of $PA \times PB$ (the power) would be different for different lines ℓ? Well...

Theorem 3.5. *Given a circle Γ, a point P and two lines ℓ_1, ℓ_2 through P that intersect Γ at A, B and A', B' respectively, we find that*

$$PA \times PB = PA' \times PB'.$$

Essentially, the power of point P with respect to Γ is the same no matter what line ℓ you draw! This includes the case where ℓ is a tangent.

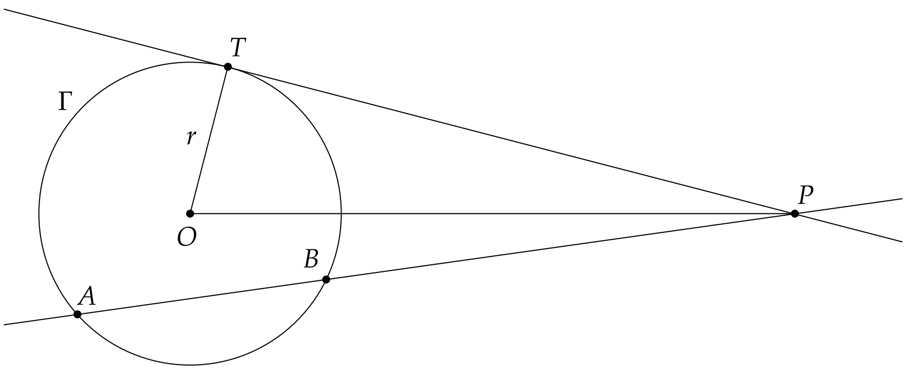
I shall leave the proof as an exercise. This is a lot easier than it sounds. [Hint: similar triangles]

Theorem 3.6. *Given the following diagram, the power of point P with respect to Γ is equal to*

$$PA \times PB = PT^2 = |PO^2 - r^2|$$

where O is the centre of the circle and r is the radius of the circle. Note, the use of absolute values because of the two cases (inside and outside the circle).

P outside the circle



Again, this is left as an exercise. Please solve this!

3.2.1 Example Problem

I consider power of a point to be really important. For this reason, I include an example problem to give you a flavour of how to use power of a point in proofs.

Problem 1. Two circles are tangent to line AB at A and B respectively (on the same side). The two circles intersect at P and Q. Prove that PQ bisects AB.

Proof. Extend PQ to intersect AB at point K. The power of K with respect to the circle tangent at A is

$$KA^2 = KP \times KQ.$$

Also, the power of K with respect to the circle tangent at B is

$$KB^2 = KP \times KQ.$$

This of course implies

$$KA^2 = KP \times KQ = KB^2.$$

Thus,

$$KA = KB.$$

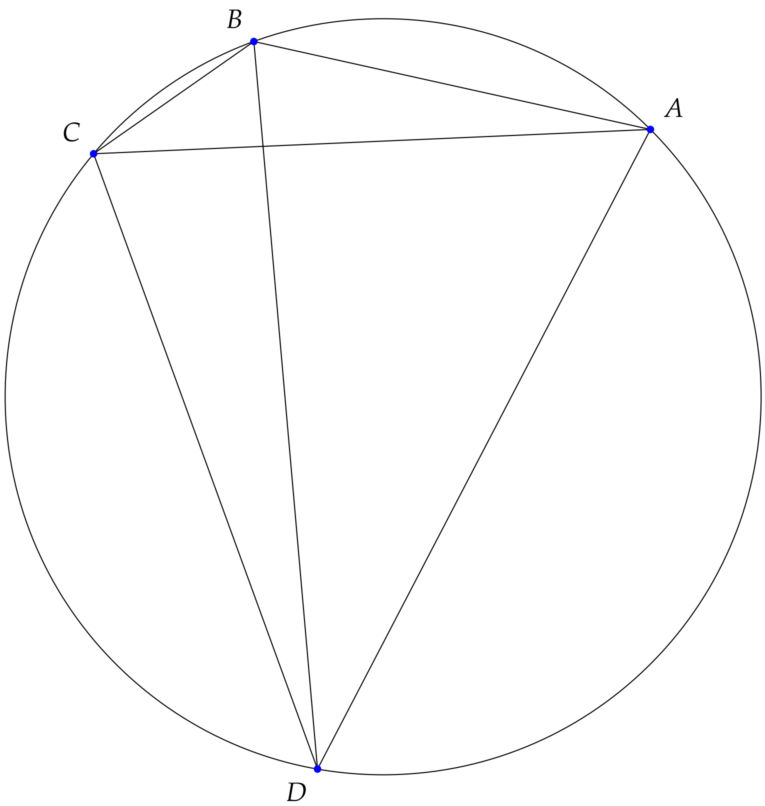
Hence, PQ bisects line segment AB at point K. □

3.3 Ptolemy's Theorem

Ptolemy's Theorem is a length theorem about cyclic quadrilaterals which we have not seen yet.

Theorem 3.7. (Ptolemy's Theorem) *The product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of opposite sides of the quadrilateral. Using the diagram below, this means*

$$AC \times BD = AB \times CD + AD \times BC.$$

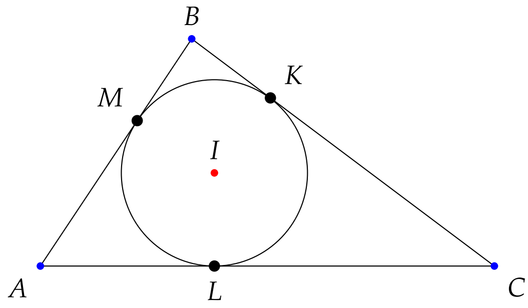


The problems section will guide you through a proof of this theorem.

3.4 Incircle Substitution

When you are working with an incircle and angle bisectors there is a useful set up to use.
As we are well, aware the standard terminology is to call $BC = a$, $AC = b$ and $AB = c$. Also, denote the semiperimeter $\frac{1}{2}(a + b + c) = s$. Then, using the notation in the below diagram,

$$AM = AL = s - a$$
$$BM = BK = s - b$$
$$CK = CL = s - c$$



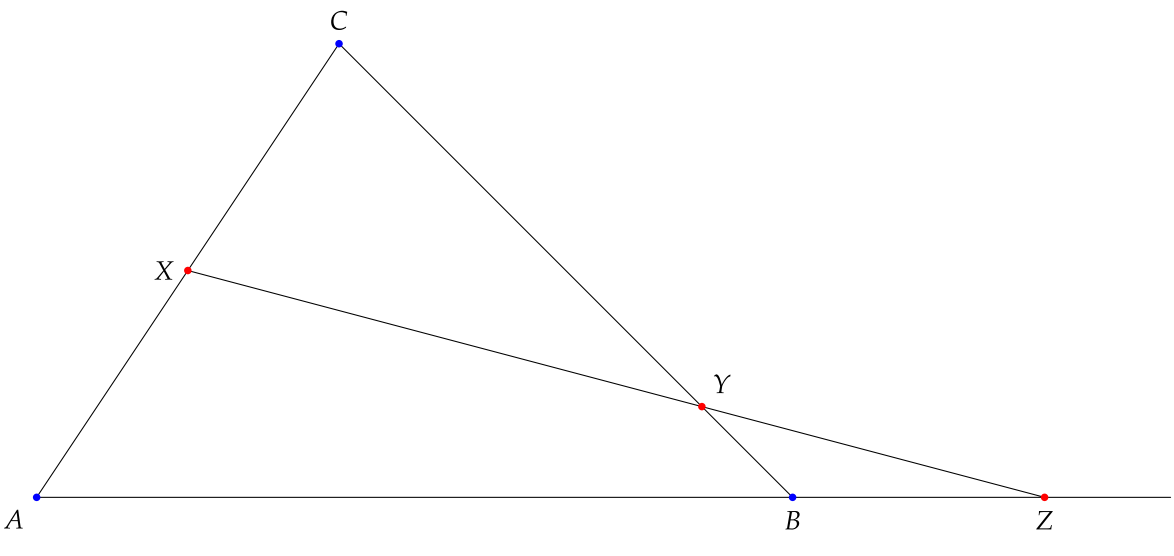
3.5 Menelaus' Theorem

Menelaus' Theorem is very closely related to Ceva's Theorem. Whereas Ceva deals with concurrent lines (lines that meet at a point), Menelaus deals with collinear points (points that line on a line). From a higher mathematical perspective (in projective geometry), these two theorems are duals of each other.

Theorem 3.8. (Menelaus' Theorem) *Let ABC be a triangle and X, Y, Z points on the sides AC, BC, AB (possibly extended) which are collinear. Then taking directed lengths,*

$$\frac{AZ}{ZB}\frac{BY}{YC}\frac{CX}{XA} = -1$$

You can of course ignore the minus sign, because directed lengths are barely ever used. Directed lengths would be familiar to those who study physics. On lines that are parallel if you choose a particular direction to be positive then the other direction is negative. For example, if one has A, B, C on a line in that order and we designate positive to be from A towards the other points, then AB, AC, BC are all positive, while CA, CB, BA are all negative.
The problems section will guide you through a proof of this theorem.



Problems

3.6 Easy

1. Prove the two theorems in the Power of a Point section!
2. Let AD, BE, DF be the altitudes of $\triangle ABC$. Let H be the orthocentre. Let M be the midpoint of BC. Prove that ME and MF are tangent to the circumcircle of $\triangle AEF$.
3. Circles Γ_1, Γ_2 intersect at A and B. Let C and D lie on Γ_1. Let lines CB and DB intersect Γ_2 again at E and F. Prove $\triangle ACD \sim \triangle AEF$.
4. Proving Ptolemy's Theorem:
   (a) Construct E on diagonal BD such that $\angle DCA = \angle ECB$. Prove that $\triangle DCA \sim \triangle ECB$.
   (b) Prove that $\triangle DCE \sim \triangle ABC$.
   (c) Write out the length ratios stemming from the similar triangles.
   (d) Remembering that $DE + EB = DB$, prove Ptolemy's Theorem.
5. Proving Menelaus' Theorem: (we shall ignore directed lengths)
   (a) Using the diagram used in the notes, extend CB to D where $AC \| DZ$. Show that $\triangle ACB \sim \triangle ZDB$ and $\triangle XCY \sim \triangle ZDY$.
   (b) Write out the similar triangle length ratios. Replace every YD with $YB + BD$, AC with $AX + XC$, CB with $CY + YB$ and AB with $AZ - BZ$.
   (c) You have all the ingredients to prove Menelaus' Theorem. Hint: write down what ZD equals from both ratio equations and equate these two. Then replace all mention of BD with ratios that only include the side lengths we are interested in. Multiply out all denominators and a lot of things should cancel.
6. Prove the incircle substitution result.

3.7 Hard

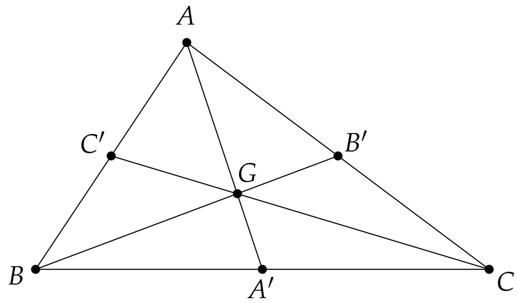
1. Let AB be the diameter of circle Γ. Let C and D be points on the tangent of Γ at point B such that B is between C and D. Let AC and AD intersect Γ again at P and Q. Prove $CPQD$ is cyclic.
2. Let the incircle of $\triangle ABC$ meet BC, AC, AB at D, E, F respectively. Let l be a line through A parallel to BC. Extend DE to meet l at J and extend DF to meet l at K. Prove A is the midpoint of JK.
3. Let H be the orthocentre of $\triangle ABC$, M be the midpoint of BC. Let D be the point diametrically opposite from A on circumcircle of ABC. Prove that H, M, D are collinear.
4. Two circles are tangent to AB at A and B respectively ob the same side of AB. The two circles intersect at P and Q. Let the two common tangents of the two circles meet at T. Prove that PT and QT are tangents to the circumcircles of $\triangle APB$ and $\triangle AQB$ respectively.
5. Let T, T_a, T_b, T_c be the points of tangency of edge AB with the incircle and three excircles of $\triangle ABC$.
   - Obviously the points T_b, A, T, T_c, B, T_a are collinear, determine all pairwise distances amongst them.
   - Determine the ratio of the inradius to an exradius, in terms of the sidelengths of $\triangle ABC$.
   - Deduce that
   $$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

4 Centres of Triangles

You have definitely heard of the centre of a circle, but did you know that a triangle also has a centre? In fact it has at least five. In this handout I will introduce the four most important ones: the centroid, the incentre, the circumcentre and the orthocentre.

4.1 Medians and the centroid

Definition. A *median* is a line segment that starts at a vertex of a triangle and travels to the midpoint of the opposite side. Note, this means medians are always on the interior of the triangle! It is a theorem of medians, that all three medians of a triangle are concurrent (they pass through the same point). This point is called the *centroid*, usually denoted G.



In order to give you an example of working with altitudes and the orthocentre, I now prove that the centroid actually exists.

Theorem 4.1. *The three medians of a triangle all pass through a single point called the centroid.*

Proof. Let the triangle be ABC. Consider the medians from A and B and let the midpoints of BC and AC be A' and B' respectively. As the medians AA' and BB' are just two lines, they definitely intersect in a single point. Let this point be called G. Extend CG to intersect AB at P. Therefore, we want to prove P is the midpoint of AB.
Notice that, because $BA' = CA'$ and the triangles $BA'A$ and $CA'A$ have the same height from their bases BA' and CA', we get

$$Area(\triangle BA'A) = Area(\triangle CA'A).$$

Similarly,

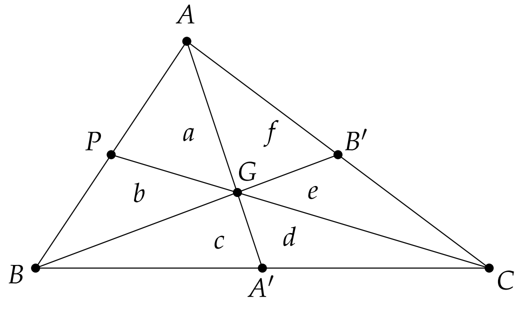
$$Area(\triangle BA'G) = Area(\triangle CA'G).$$

If we focus on $AB' = CB'$, we get similar conclusions

$$Area(\triangle BB'A) = Area(\triangle BB'C).$$

and

$$Area(\triangle GB'A) = Area(\triangle GB'C).$$



To make things easier, call $Area(\triangle APG) = a$, $Area(\triangle BPG) = b$, $Area(\triangle BA'G) = c$, $Area(\triangle CA'G) = d$, $Area(\triangle CB'G) = e$, $Area(\triangle AB'G) = f$.
Then the four equations above become:

$$
\begin{aligned}
a + b + c &= d + e + f \\
c &= d \\
a + b + f &= c + d + e \\
e &= f
\end{aligned}
$$

Solving these four equations actually gives us $c = d = e = f$. Now consider

$$\frac{Area(\triangle ACP)}{Area(\triangle PCB)} = \frac{AP}{PB} \qquad \text{and} \qquad \frac{Area(\triangle AGP)}{Area(\triangle PGB)} = \frac{AP}{PB}.$$

Inputting what we know about the areas gives

$$\frac{a + 2c}{b + 2c} = \frac{AP}{PB} = \frac{a}{b}.$$

Cross-multiplying, subtracting ab and dividing by $2c \neq 0$ yields

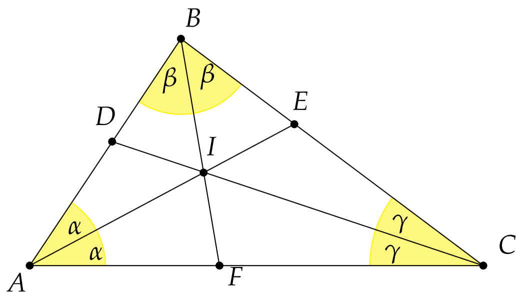
$$a = b.$$

So, $Area(\triangle APG) = Area(\triangle BPG)$. Given that triangles APG and BPG have the same height from AB to G, yields that $AP = BP$, so P is the midpoint of AB as required. □

Note, that the centroid is actually also known as the centre of gravity (hence why it is denoted G). If the triangle was an actual object, then the centroid would be the point under which a pin could be placed to balance the whole triangle.

4.2 Angle bisectors and the incentre

Definition. The *internal angle bisector* of an angle is a line that divides the angle in half. In a triangle, which has three angles, we therefore, have three angle bisectors. It is a theorem of angle bisectors, that all three internal angle bisectors of a triangle are concurrent (they pass through the same point). This point is called the *incentre*, usually denoted I.



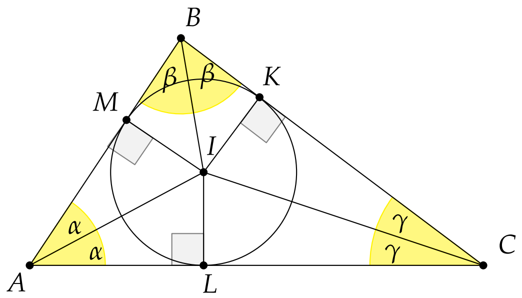
What is the incentre, the centre of? We will see this a little bit later on. First, does it really exist?

Theorem 4.2. *The three internal angle bisectors of a triangle all pass through a single point called the incentre.*

I shall leave the proof as an exercise. Time to stretch those geometry muscles we have been working on.

Theorem 4.3. *The incentre is the centre of the circle that touches all three sides of the triangle.*

Proof. Drop the perpendiculars from I to the sides AB, AC and BC as shown in the below diagram.



Because $\angle IMA = \angle ILA = 90°$, $\angle MAI = \angle IAL$ and AI is common, we have that

$$\triangle MAI \cong \triangle LAI.$$

Therefore, $MI = LI$. Similarly, one can show $\triangle MBI \cong \triangle KBI$ implying $MI = KI$.
Hence,

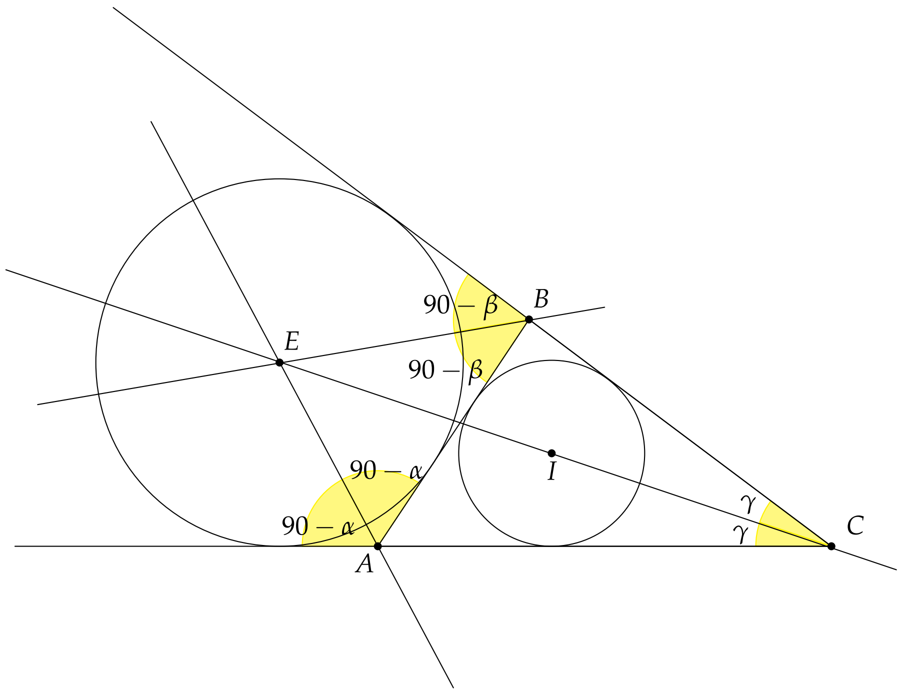
$$KI = MI = LI.$$

Thus, I is the centre of the circle passing through K, L and M. Since the sides of the triangle are at right angles to the radii KI, LI and MI, the circle touches the sides of the triangle. □

Definition. The circle that touches all three sides of a triangle internally is called the *incircle*. Its centre is the incentre described above.

There is also the concept of external angle bisectors and excentres.

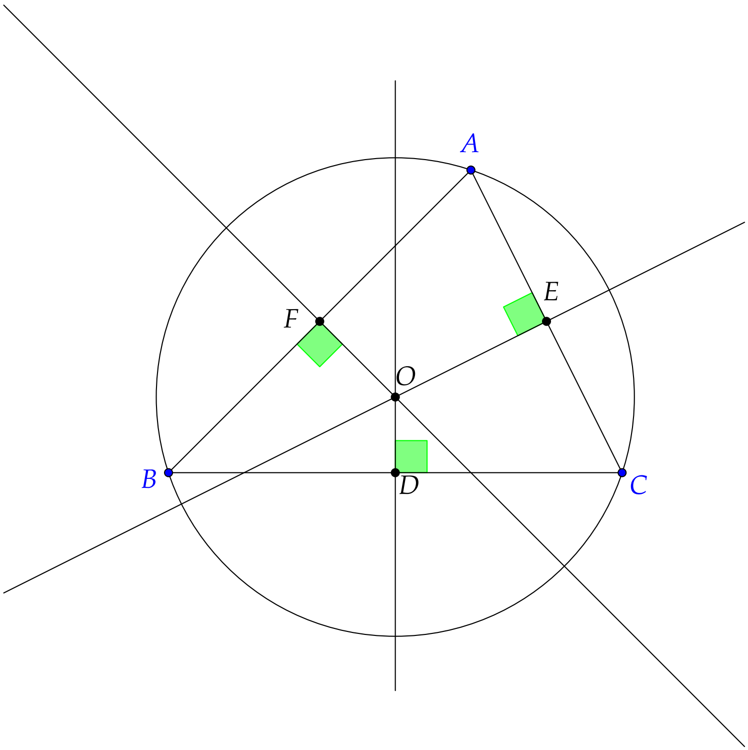
Definition. An *external angle bisector* of $\angle ABC$ is the line that is at right angles to the internal angle bisector and passes through B. If one was to extend the ray AB and bisect the angle formed by the lines CB and the ray AB extended beyond B, this would coincide with the external angle bisector. It turns out that any two external angle bisectors of a triangle and the internal angle bisector of the remaining angle meet in a point called the *excentre*, which is the centre of the *excircle*, a circle touching one of the sides of the triangle as well as the extensions of the remaining two sides.



4.3 Perpendicular bisectors and the circumcentre

In this section, we explore perpendicular bisectors and the circumcentre. While this will include many right angles and lots of midpoints, because perpendicular bisectors are not cevans (line segments from a vertex to the opposite side), this will be completely different!

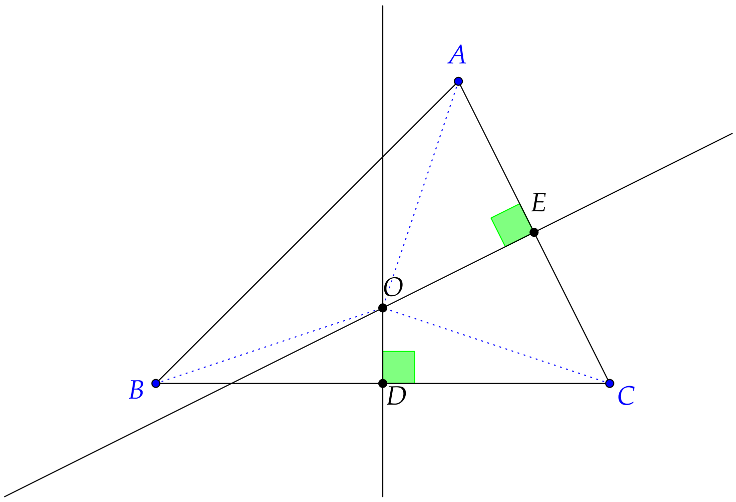
Definition. In general, the *perpendicular bisector* of a line segment is a line that is perpendicular to it and passes through the midpoint of the line segment. Because a triangle has three sides, one can draw the perpendicular bisector of each side. It is a theorem of perpendicular bisectors, that all three perpendicular bisectors of a triangle are concurrent (they pass through the same point). This point is called the *circumcentre*, usually denoted O.



As is implied by the previous diagram, the circumcentre has something to do with the circumcircle of the triangle. See the following theorem for how they relate. (Have a guess first!)

Theorem 4.4. *The three perpendicular bisectors of a triangle all pass through a single point called the circumcentre. It is the centre of the circumcircle of the triangle (the circle that passes through the three vertices).*

Proof. First, we show that any two perpendicular bisectors of a triangle meet at the circumcentre (the centre of the circumcircle). Let the triangle be ABC. Consider the perpendicular bisectors of sides AC and BC (whose midpoints are E and D respectively). Let them meet at point O.



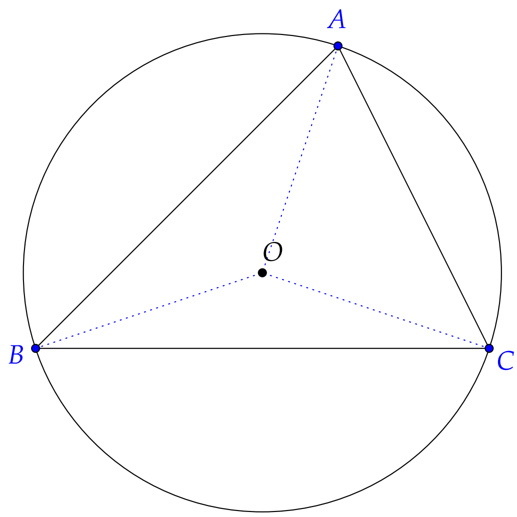
Because $AE = CE$, $\angle AEO = \angle CEO = 90^\circ$ and EO is common, we have that

$$\triangle AEO \cong \triangle CEO.$$

Therefore, $AO = CO$. Similarly, one can show $\triangle BDO \cong \triangle CDO$ implying $BO = CO$.
Hence,

$$AO = BO = CO.$$

Thus, O is the centre of the circle passing through A, B and C. This proves also that the point of intersection is independent of which two perpendicular bisectors were chosen. So, if we investigated the perpendicular bisectors of AB and AC instead, we would find their point of intersection would also be the centre of the circumcircle, which is unique. Hence, all three perpendicular bisectors pass through the same point O, which is also the circumcentre.



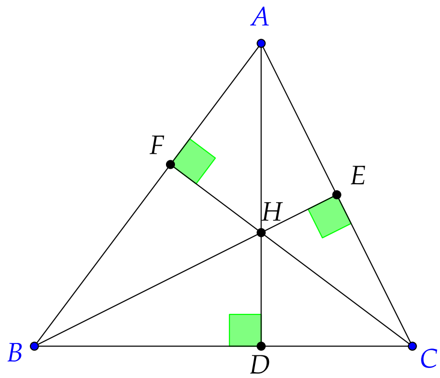
□

Note, that the circumcentre can actually be outside the triangle. The radius of the circumcircle is usually denoted by R.

4.4 Altitudes and the orthocentre

This section focusses on altitudes and the orthocentre.

Definition. An *altitude* is a line segment that starts at a vertex of a triangle and travels perpendicular to the opposite stopping when it reaches that side. Note, that the altitude can be completely outside the triangle!!! Call the points of intersection of the altitudes with the opposite side the *feet of the altitude*. It is a theorem of altitudes, that all three altitudes of a triangle are concurrent (they pass through the same point). This point is called the *orthocentre* usually denoted H.

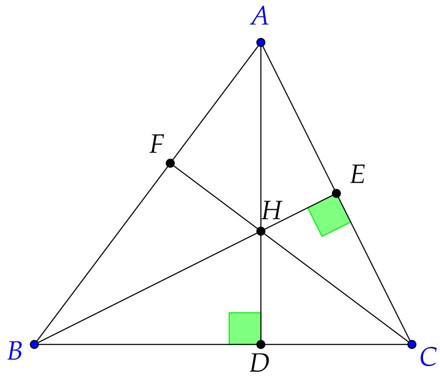


In order to give you an example of working with altitudes and the orthocentre, I now prove that the orthocentre actually exists.

Theorem 4.5. *The three altitudes of a triangle all pass through a single point called the orthocentre.*

Proof. Let the triangle be ABC. Consider the altitudes from A and B and let their feet be called D and E respectively. As these are just two lines, they definitely intersect in a single point. Let this point be called H. Therefore, we have $CDHE$ is cyclic, by the fact that $\angle CDH = \angle CEH = 90°$. Similarly, $AEDB$ is cyclic, because $\angle AEB = \angle ADB = 90°$. Therefore, using angles subtended by the same arc, we get

$$\angle ECH = \angle EDH = \angle EBA$$



If we extend CH to meet AB in F, we therefore get that $ECBF$ is cyclic (because $\angle ECF = \angle EBF$). Hence, $\angle CFB = \angle CEB = 90°$. Thus, CF is an altitude and we are done. □

The triangle formed by the feet of the altitudes has a special name.

Definition. The *pedal* (or *orthic*) triangle is formed by joining the three points which are the feet of the altitudes.

Problems

4.5 Easy

4.5.1 Centroid

In these centroid problems, assume the triangle is ABC with medians AA', BB' and CC'. The centroid is called point G.

1. When is a median also an altitude (line segment from a vertex perpendicular to the opposite side)?
2. When is the median the same length as an altitude?
3. [Midline Theorem] Prove that $A'B'$ is parallel to AB and half the length. Conversely, prove that if $A'P$ is parallel to AB where P lies on AC, then $P = B'$.
4. What is the area of $\triangle A'B'C'$ in terms of the area of $\triangle ABC$?

4.5.2 Incentre

In these incentre problems, assume the triangle is ABC. Assume the three internal angle bisectors are AD, BE, CF where D is on BC, E is on AC and F is on AB.

1. Does the perpendicular to the side BC at the point D go through the incentre? Does AI contain D?
2. If AI meets the circumcircle in U, show that $OU \perp BC$.
3. Show that in an equilateral triangle, the inradius is half the circumradius.
4. Show that $AF + AE = AB - BC + CA$.
5. Show that the three internal angle bisectors of a triangle concur.

4.5.3 Circumcentre

In these circumcentre problems, assume the triangle is ABC where AB, AC and BC have midpoints F, E and D respectively.

1. When is the circumcentre the same point as the orthocentre (intersection of the altitudes)? When is it the same point as the centroid?
2. Where is the circumcentre of a right angled triangle? Where is the circumcentre of an equilateral triangle? Isosceles triangle?
3. In triangle ABC, O is the circumcentre, H is the orthocentre. Show that $\angle ABH = \angle CBO$.
4. What is the circumradius in terms of the sides of the triangle for an equilateral triangle?
5. If the circumcentre of a triangle lies on one of the sides, what kind of triangle is it?

4.5.4 Orthocentre

In these orthocentre problems, assume the triangle is ABC with altitudes AD, BE and CF. The orthocentre is called point H. Extend CH to intersect with side AB (possibly extended).

1. Show that $\angle ABH = \angle CBO$.
2. Show that $\angle BHC + \angle A = 180^\circ$.
3. If the angles in ABC are α, β, γ, what are the angles of the pedal triangle?
4. Show that A is the orthocentre of triangle BCH. Show that similar results hold for B and C.

4.6 Hard

4.6.1 Centroid

In these centroid problems, assume the triangle is ABC with medians AA', BB' and CC'. The centroid is called point G.

1. Prove that $\angle A'AC = \angle ABB'$ if and only if $\angle AC'C = \angle AA'B$.
2. What is the ratio $AG : GA'$?
3. [Apollonius' Theorem] Prove that
$$AB^2 + AC^2 = \frac{BC^2}{2} + 2A'A^2.$$
4. Determine, with proof, the position of the point P in the plane of ABC such that $AP \times AG + BP \times BG + CP \times CG$ is a minimum, and express this minimum value in terms of the side lengths of $\triangle ABC$.

4.6.2 Incentre

In these incentre problems, assume the triangle is ABC. Assumes the three internal angle bisectors are AD, BE, CF where D is on BC, E is on AC and F is on AB.

1. Show that the distance between D and the midpoint of BC is $\frac{1}{2}|b - c|$.
2. Show that the radius of the inscribed circle of a right triangle is equal to half the difference between the hypotenuse and the sum of the other two sides.
3. Show how to find the incentre of a triangle, and construct its incircle, with ruler and compass.
4. Points I, I_A are the incentre and excentre opposite A in $\triangle ABC$. Prove that $BICI_A$ is cyclic. Furthermore prove that the circumcentre of $BICI_A$ lies on the circumcircle of $\triangle ABC$.
5. If the line segment AI meets the incircle in P, show that P is the incentre of triangle AEF.

4.6.3 Circumcentre

In these circumcentre problems, assume the triangle is ABC where AB, AC and BC have midpoints F, E and D respectively.

1. Show that the triangle formed by the tangents at A, B, C to the circumcircle is similar to the pedal triangle.
2. Show that AO, AH are equally inclined to the bisector of angle A.
3. An acute-angled triangle ABC is inscribed in a circle with centre D. The circle drawn through A, B and D intersects AC and BC again at M and N respectively. Prove that the circumradii of triangles ABD and MNC are equal.

4.6.4 Orthocentre

In these orthocentre problems, assume the triangle is ABC with altitudes AD, BE and CF. The orthocentre is called point H. Extend CH to intersect with side AB (possibly extended).

1. Show that ABC, ABH, ACH and BCH all have the same pedal triangle.
2. Show that $BD \times DC = AD \times HD$. Hence, show that $AH \times HD = BH \times HE = CH \times HF$.
3. Through A, B, C draw lines parallel to BC, CA, AB respectively, forming a triangle KLM. By considering the circumcentre of triangle KLM, show that the altitudes of triangle ABC are concurrent.

5 Theorems in Triangles

This handout is an opportunity to learn some more theorems, but perhaps more importantly shore up the information that we have been learning.

5.1 Pythagoras

This is a very well known theorem in geometry that I include for completeness' sake.

Theorem 5.1. *In a right angled triangle ABC right angled at C, the side lengths satisfy the fomula*

$$a^2 + b^2 = c^2.$$

5.2 Angle Bisector Theorem

When dealing with angle bisectors, here is a handy theorem to have.

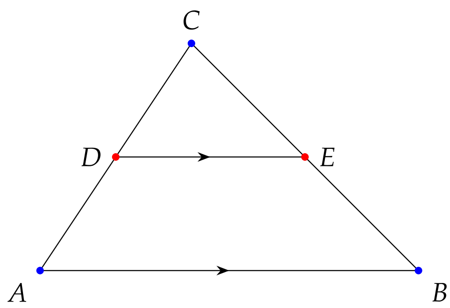
Theorem 5.2. *In $\triangle ABC$, draw an angle bisector from A to meet BC at D (D is clearly on the interior of BC). Then*

$$\frac{BD}{DC} = \frac{BA}{AC}.$$

5.3 Midline Theorem

We have used this theorem before, but I state it now officially.

Theorem 5.3. *The line joining the midpoints of AB and AC is parallel to BC and half its length. The converse is also true: the line segment DE with D on AB and E on AC that is parallel to BC and half its length actually joins the midpoints of AB and AC (ie. D is midpoint of AB and E is the midpoint of AC).*



5.4 Ceva's Theorem

This is a much more advanced theorem than the other three above. It is really useful when you are working with random line segments that intersect inside a triangle.

Definition. A *cevan* is a line segment inside a triangle that starts at a vertex and ends on the opposite side.

Examples of cevans include an angle bisector, median, altitude, but a perpendicular bisector is almost never a cevan! We can now state Ceva's Theorem.

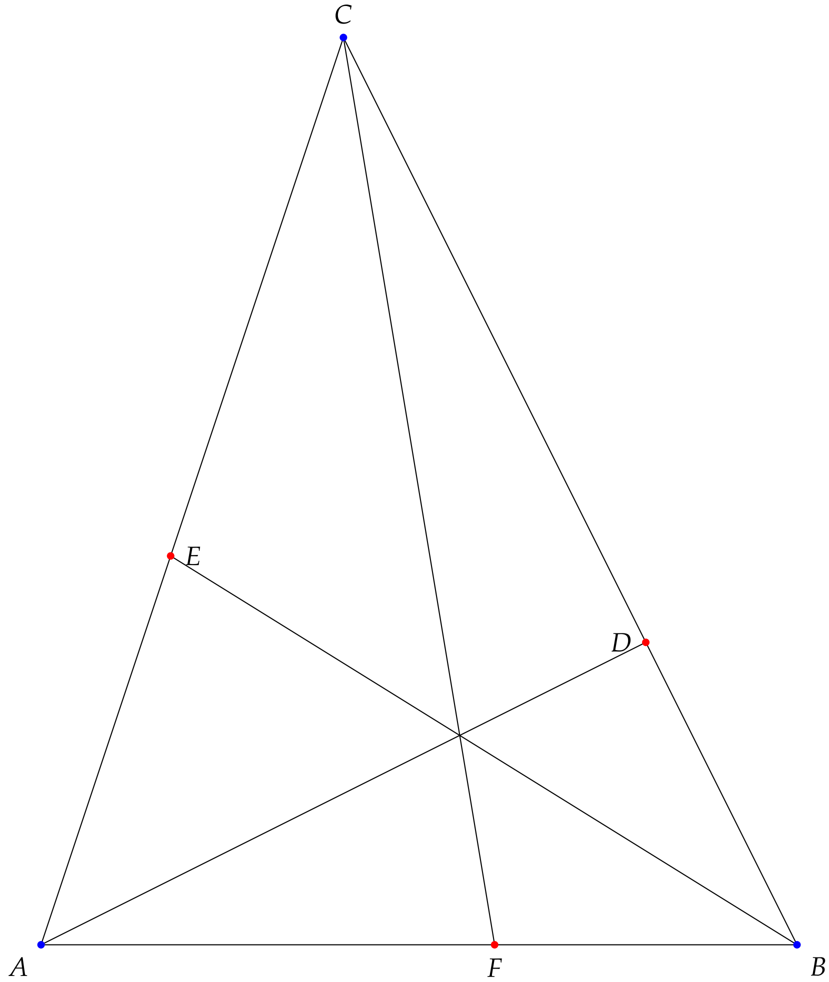
Theorem 5.4. *In $\triangle ABC$, if the cevans AD, BE, CF are concurrent, then*

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Moreover, the converse is true: if the cevans AD, BE, CF satisfy

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1,$$

then they are concurrent.



5.5 Nine-Point Circle

This is a much lesser known result about circles in triangles and this is the fifth centre of the triangle I mentioned in the previous handout.

Theorem 5.5. *The midpoint of the three sides, the midpoints of the lines joining the orthocentre to the three vertices, and the feet of the three altitudes, all lie on a circle, called the* nine-point circle*.*

In fact, there is more to it:

Theorem 5.6. *The nine-point centre,* N*, lies on a line called the* Euler line *which also contains the orthocentre, centroid and circumcentre. In fact, they occur in the order H, N, G, O with*

$$HN : NG : GO = 3 : 1 : 2.$$

I leave the drawing and proving of this as a worthwhile exercise.

Problems

5.6 Easy

1. When is a perpendicular bisector a cevan?

2. Prove that the medians are concurrent using Ceva's Theorem.

3. Prove that the angle bisectors are concurrent using Ceva's Theorem.

4. In acute $\triangle ABC$, draw an altitude AD from A to BC.

   (a) Show that $\frac{BD}{CD} = \frac{AB}{AC} \times \frac{\cos B}{\cos C}$.

   (b) Hence, show that the altitudes are concurrent using Ceva's Theorem.

   (c) How would you amend the proof to make it work for an obtuse triangle?

5. In triangle ABC, D, E, F are feet of the altitudes opposite A, B, C respectively. H is the orthocentre. Show that
$$AH \times HD = BH \times HE = CH \times HF.$$

6. Prove that if the incircle of $\triangle ABC$ touches BC, AC, AB at X, Y, Z respectively, then AX, BY, CZ are concurrent.

5.7 Hard

1. [Euler Line] Prove that the orthocentre H, the centroid G, and the circumcentre O of a triangle are collinear. Hence show that $HG = 2GO$.

2. Let H be the orthocentre of triangle ABC. Let A', B', C' be the reflections of H about sides AB, BC, CA respectively. Prove that the six points (A, B, C, A', B', C') all lie on a circle.

3. In the acute angled triangle ABC, let D be the foot of the perpendicular from A to BC, let E be the foot of the perpendicular from D to AC and let F be a point on the line segment DE. Prove that $AF \perp BE$ if and only if
$$\frac{FE}{FD} = \frac{BD}{DC}.$$

4. Let A, B, C be three collinear points on line ℓ. Pick an arbitrary point P in the plane, and another arbitrary point Q on line BP. Let AQ intersect CP at X, and CQ intersect AP at Y. Prove that, regardless of the choice of P and Q, all the possible lines XY are concurrent at a point on ℓ.

5. Let A, B, C be three collinear points. Find all points X in the plane such that $AX : CX = AB : CB$.

6. Points D, E and F are chosen on the sides BC, AC and AB respectively, so that $FD = BD$ and $ED = CD$. Prove that the circumcentre of triangle AFE lie on the bisector of $\angle FDE$.