CHAPTER 1

Angle Chasing

> This is your last chance. After this, there is no turning back. You take the blue pill—the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill—you stay in Wonderland and I show you how deep the rabbit-hole goes.
> Morpheus in *The Matrix*

Angle chasing is one of the most fundamental skills in olympiad geometry. For that reason, we dedicate the entire first chapter to fully developing the technique.

1.1 Triangles and Circles

Consider the following example problem, illustrated in Figure 1.1A.

Example 1.1. In quadrilateral $WXYZ$ with perpendicular diagonals (as in Figure 1.1A), we are given $\angle WZX = 30^\circ$, $\angle XWY = 40^\circ$, and $\angle WYZ = 50^\circ$.

(a) Compute $\angle Z$.
(b) Compute $\angle X$.

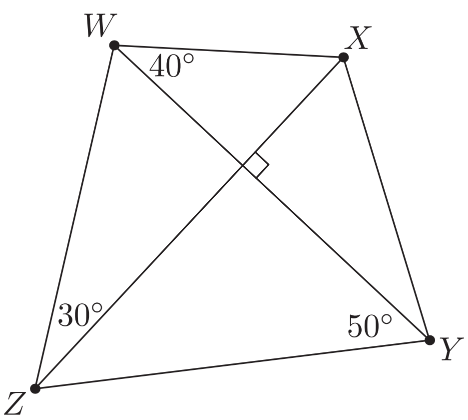

Figure 1.1A. Given these angles, which other angles can you compute?

You probably already know the following fact:

Proposition 1.2 (Triangle Sum). *The sum of the angles in a triangle is* 180°.

As it turns out, this is not sufficient to solve the entire problem, only the first half. The next section develops the tools necessary for the second half. Nevertheless, it is perhaps surprising what results we can derive from Proposition 1.2 alone. Here is one of the more surprising theorems.

Theorem 1.3 (Inscribed Angle Theorem). *If $\angle ACB$ is inscribed in a circle, then it subtends an arc with measure $2\angle ACB$.*

Proof. Draw in $\overline{OC}$. Set $\alpha = \angle ACO$ and $\beta = \angle BCO$, and let $\theta = \alpha + \beta$.

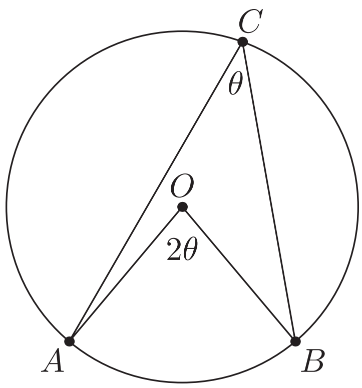

Figure 1.1B. The inscribed angle theorem.

We need some way to use the condition $AO = BO = CO$. How do we do so? Using isosceles triangles, roughly the only way we know how to convert lengths into angles. Because $AO = CO$, we know that $\angle OAC = \angle OCA = \alpha$. How does this help? Using Proposition 1.2 gives

$$\angle AOC = 180^\circ - (\angle OAC + \angle OCA) = 180^\circ - 2\alpha.$$

Now we do exactly the same thing with B. We can derive

$$\angle BOC = 180^\circ - 2\beta.$$

Therefore,

$$\angle AOB = 360^\circ - (\angle AOC + \angle BOC) = 360^\circ - (360^\circ - 2\alpha - 2\beta) = 2\theta$$

and we are done. □

We can also get information about the centers defined in Section 0.2. For example, recall the *incenter* is the intersection of the angle bisectors.

Example 1.4. If I is the incenter of $\triangle ABC$ then

$$\angle BIC = 90^\circ + \frac{1}{2}A.$$

Proof. We have

$$\begin{aligned}\angle BIC &= 180^\circ - (\angle IBC + \angle ICB)\\ &= 180^\circ - \frac{1}{2}(B + C)\\ &= 180^\circ - \frac{1}{2}(180^\circ - A)\\ &= 90^\circ + \frac{1}{2}A.\end{aligned}$$

□

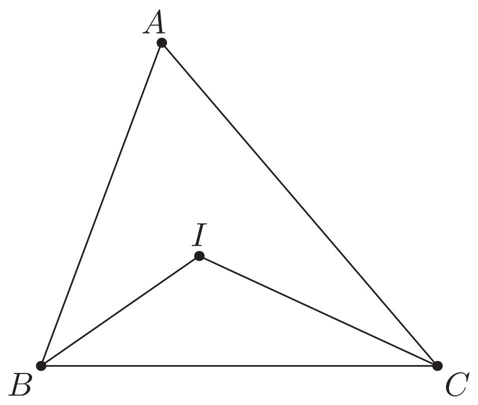

Figure 1.1C. The incenter of a triangle.

Problems for this Section

Problem 1.5. Solve the first part of Example 1.1. **Hint**: 185

Problem 1.6. Let ABC be a triangle inscribed in a circle ω. Show that $\overline{AC} \perp \overline{CB}$ if and only if $\overline{AB}$ is a diameter of ω.

Problem 1.7. Let O and H denote the circumcenter and orthocenter of an acute $\triangle ABC$, respectively, as in Figure 1.1D. Show that $\angle BAH = \angle CAO$. **Hints**: 540 373

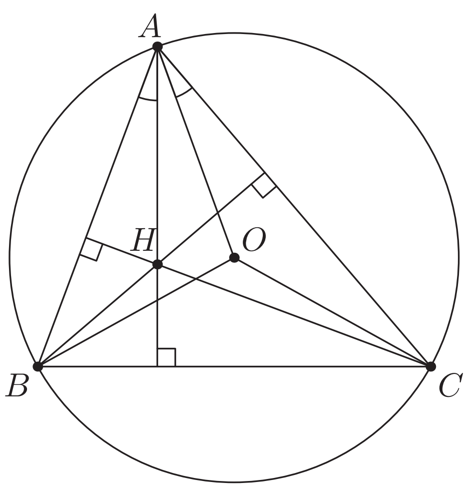

Figure 1.1D. The orthocenter and circumcenter. See Section 0.2 if you are not familiar with these.

1.2 Cyclic Quadrilaterals

The heart of this section is the following proposition, which follows directly from the inscribed angle theorem.

Proposition 1.8. *Let $ABCD$ be a convex cyclic quadrilateral. Then $\angle ABC + \angle CDA = 180^\circ$ and $\angle ABD = \angle ACD$.*

Here a **cyclic quadrilateral** refers to a quadrilateral that can be inscribed in a circle. See Figure 1.2A. More generally, points are **concyclic** if they all lie on some circle.

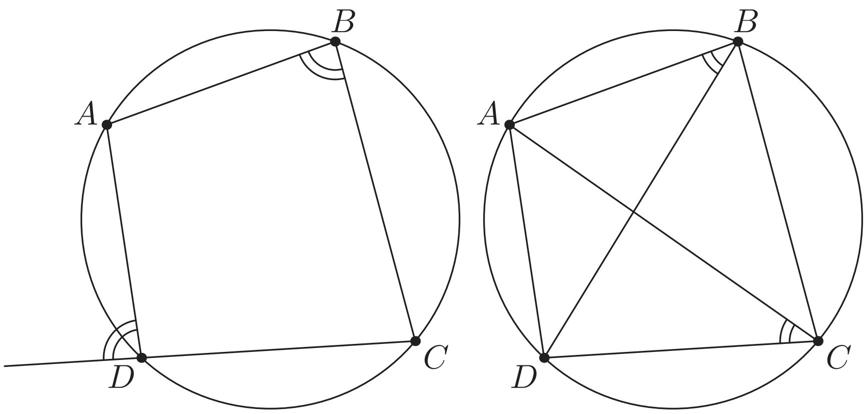

Figure 1.2A. Cyclic quadrilaterals with angles marked.

At first, this result seems not very impressive in comparison to our original theorem. However, it turns out that the converse of the above fact is true as well. Here it is more explicitly.

Theorem 1.9 (Cyclic Quadrilaterals). *Let $ABCD$ be a convex quadrilateral. Then the following are equivalent:*

(i) $ABCD$ *is cyclic.*
(ii) $\angle ABC + \angle CDA = 180^\circ$.
(iii) $\angle ABD = \angle ACD$.

This turns out to be extremely useful, and several applications appear in the subsequent sections. For now, however, let us resolve the problem we proposed at the beginning.

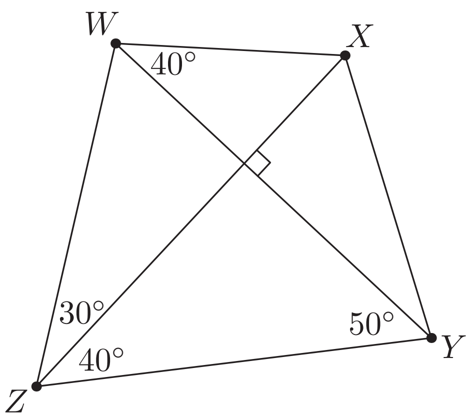

Figure 1.2B. Finishing Example 1.1. We discover $WXYZ$ is cyclic.

Solution to Example 1.1, part (b). Let P be the intersection of the diagonals. Then we have $\angle PZY = 90^\circ - \angle PYZ = 40^\circ$. Add this to the figure to obtain Figure 1.2B.

Now consider the 40° angles. They satisfy condition (iii) of Theorem 1.9. That means the quadrilateral $WXYZ$ is cyclic. Then by condition (ii), we know

$$\angle X = 180^\circ - \angle Z$$

Yet $\angle Z = 30^\circ + 40^\circ = 70^\circ$, so $\angle X = 110^\circ$, as desired. □

In some ways, this solution is totally unexpected. Nowhere in the problem did the problem mention a circle; nowhere in the solution does its center ever appear. And yet, using the notion of a cyclic quadrilateral reduced it to a mere calculation, whereas the problem was not tractable beforehand. This is where Theorem 1.9 draws its power.

We stress the importance of Theorem 1.9. It is not an exaggeration to say that more than 50% of standard olympiad geometry problems use it as an intermediate step. We will see countless applications of this theorem throughout the text.

Problems for this Section

Problem 1.10. Show that a trapezoid is cyclic if and only if it is isosceles.

Problem 1.11. Quadrilateral $ABCD$ has $\angle ABC = \angle ADC = 90^\circ$. Show that $ABCD$ is cyclic, and that $(ABCD)$ (that is, the circumcircle of $ABCD$) has diameter $\overline{AC}$.

1.3 The Orthic Triangle

In $\triangle ABC$, let D, E, F denote the feet of the altitudes from A, B, and C. The $\triangle DEF$ is called the **orthic triangle** of $\triangle ABC$. This is illustrated in Figure 1.3A.

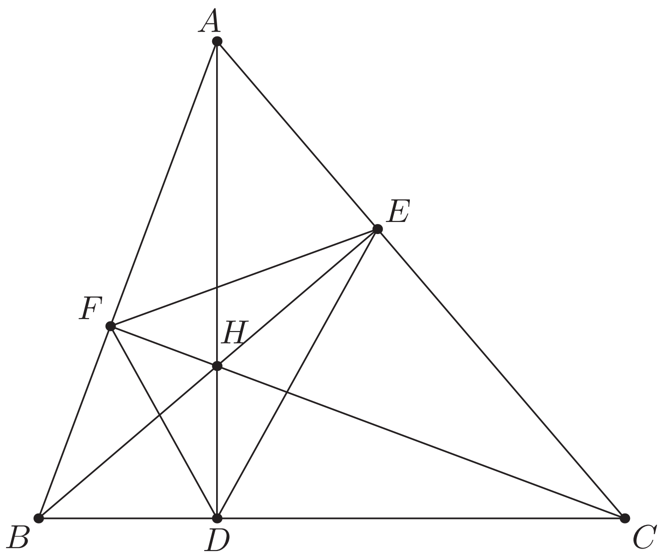

Figure 1.3A. The orthic triangle.

It also turns out that lines AD, BE, and CF all pass through a common point H, which is called the **orthocenter** of H. We will show the orthocenter exists in Chapter 3.

Although there are no circles drawn in the figure, the diagram actually contains six cyclic quadrilaterals.

Problem 1.12. In Figure 1.3A, there are six cyclic quadrilaterals with vertices in $\{A, B, C, D, E, F, H\}$. What are they? **Hint**: 91

To get you started, one of them is $AFHE$. This is because $\angle AFH = \angle AEH = 90^\circ$, and so we can apply (ii) of Theorem 1.9. Now find the other five!

Once the quadrilaterals are found, we are in a position of power; we can apply any part of Theorem 1.9 freely to these six quadrilaterals. (In fact, you can say even more—the right angles also tell you where the diameter of the circle is. See Problem 1.6.) Upon closer inspection, one stumbles upon the following.

Example 1.13. Prove that H is the incenter of $\triangle DEF$.

Check that this looks reasonable in Figure 1.3A.

We encourage the reader to try this problem before reading the solution below.

Solution to Example 1.13. Refer to Figure 1.3A. We prove that $\overline{DH}$ is the bisector of $\angle EDF$. The other cases are identical, and left as an exercise.

Because $\angle BFH = \angle BDH = 90^\circ$, we see that $BFHD$ is cyclic by Theorem 1.9. Applying the last clause of Theorem 1.9 again, we find

$$\angle FDH = \angle FBH.$$

Similarly, $\angle HEC = \angle HDC = 90^\circ$, so $CEHD$ is cyclic. Therefore,

$$\angle HDE = \angle HCE.$$

Because we want to prove that $\angle FDH = \angle HDE$, we only need to prove that $\angle FBH = \angle HCE$; in other words, $\angle FBE = \angle FCE$. This is equivalent to showing that $FBCE$ is cyclic, which follows from $\angle BFC = \angle BEC = 90^\circ$. (One can also simply show that both are equal to $90^\circ - A$ by considering right triangles BEA and CFA.)

Hence, $\overline{DH}$ is indeed the bisector, and therefore we conclude that H is the incenter of $\triangle DEF$. □

Combining the results of the above, we obtain our first configuration.

Lemma 1.14 (The Orthic Triangle). *Suppose $\triangle DEF$ is the orthic triangle of acute $\triangle ABC$ with orthocenter H. Then*

(a) Points A, E, F, H lie on a circle with diameter $\overline{AH}$.
(b) Points B, E, F, C lie on a circle with diameter $\overline{BC}$.
(c) H is the incenter of $\triangle DEF$.

Problems for this Section

Problem 1.15. Work out the similar cases in the solution to Example 1.13. That is, explicitly check that $\overline{EH}$ and $\overline{FH}$ are actually bisectors as well.

Problem 1.16. In Figure 1.3A, show that $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ are each similar to $\triangle ABC$. **Hint**: 181

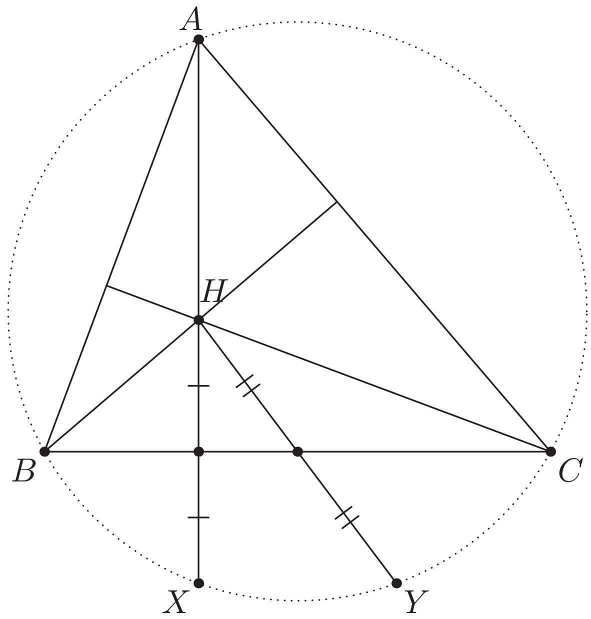

Figure 1.3B. Reflecting the orthocenter. See Lemma 1.17.

Lemma 1.17 (Reflecting the Orthocenter). *Let H be the orthocenter of $\triangle ABC$, as in Figure 1.3B. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.*

(a) Show that X lies on (ABC).
(b) Show that $\overline{AY}$ is a diameter of (ABC). **Hint**: 674

1.4 The Incenter/Excenter Lemma

We now turn our attention from the orthocenter to the incenter. Unlike before, the cyclic quadrilateral is essentially given to us. We can use it to produce some interesting results.

Lemma 1.18 (The Incenter/Excenter Lemma). *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

(a) The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$.
(b) Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.

By "exterior angle", we mean that ray BI_A bisects the angle formed by the segment BC and the extension of line AB past B. The point I_A is called the **A-excenter*** of $\triangle ABC$; we visit it again in Section 2.6.

Let us see what we can do with cyclic quadrilateral $ABLC$.

* Usually the A-excenter is defined as the intersection of exterior angle bisectors of $\angle B$ and $\angle C$, rather than as the reflection of I over L. In any case, Lemma 1.18 shows these definitions are equivalent.

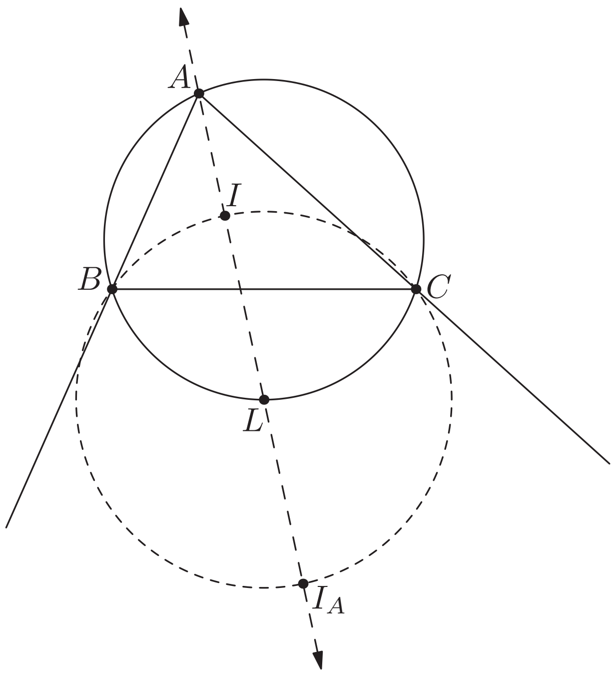

Figure 1.4A. Lemma 1.18, the incenter/excenter lemma.

Proof. Let $\angle A = 2\alpha$, $\angle B = 2\beta$, and $\angle C = 2\gamma$ and notice that $\angle A + \angle B + \angle C = 180^\circ \Rightarrow \alpha + \beta + \gamma = 90^\circ$.

Our first goal is to prove that $LI = LB$. We prove this by establishing $\angle IBL = \angle LIB$ (this lets us convert the conclusion completely into the language of angles). To do this, we invoke (iii) of Theorem 1.9 to get $\angle CBL = \angle LAC = \angle IAC = \alpha$. Therefore,

$$\angle IBL = \angle IBC + \angle CBL = \beta + \alpha.$$

All that remains is to compute $\angle BIL$. But this is simple, as

$$\angle BIL = 180^\circ - \angle AIB = \angle IBA + \angle BAI = \alpha + \beta$$

Therefore triangle LBI is isosceles, with $LI = LB$, which is what we wanted.

Similar calculations give $LI = LC$.

Because $LB = LI = LC$, we see that L is indeed the center of (IBC). Because L is given to be the midpoint of $\overline{II_A}$, it follows that $\overline{II_A}$ is a diameter of (LBC) as well.

Let us now approach the second part. We wish to show that $\angle I_ABC = \frac{1}{2}(180^\circ - 2\beta) = 90^\circ - \beta$. Recalling that $\overline{II_A}$ is a diameter of the circle, we observe that

$$\angle IBI_A = \angle ICI_A = 90^\circ.$$

so $\angle I_ABC = \angle I_ABI - \angle IBC = 90^\circ - \beta$.

Similar calculations yield that $\angle BCI_A = 90^\circ - \gamma$, as required. □

This configuration shows up very often in olympiad geometry, so recognize it when it appears!

Problem for this Section

Problem 1.19. Fill in the two similar calculations in the proof of Lemma 1.18.

1.5 Directed Angles

Some motivation is in order. Look again at Figure 1.3A. We assumed that $\triangle ABC$ was acute. What happens if that is not true? For example, what if $\angle A > 90^\circ$ as in Figure 1.5A?

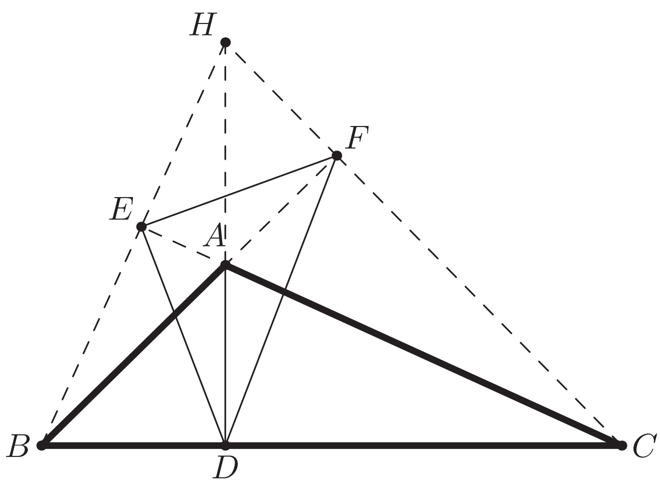

Figure 1.5A. No one likes configuration issues.

There should be something scary in the above figure. Earlier, we proved that points B, E, A, D were concyclic using criterion (iii) of Theorem 1.9. Now, the situation is different. Has anything changed?

Problem 1.20. Recall the six cyclic quadrilaterals from Problem 1.12. Check that they are still cyclic in Figure 1.5A.

Problem 1.21. Prove that, in fact, A is the orthocenter of $\triangle HBC$.

In this case, we are okay, but the dangers are clear. For example, when $\triangle ABC$ was acute, we proved that B, H, F, D were concyclic by noticing that the opposite angles satisfied $\angle BDH + \angle HFB = 180^\circ$. Here, however, we instead have to use the fact that $\angle BDH = \angle BFH$; in other words, for the same problem we have to use different parts of Theorem 1.9. We should not need to worry about solving the same problem twice!

How do we handle this? The solution is to use **directed angles** mod 180°. Such angles will be denoted with a $\measuredangle$ symbol instead of the standard $\angle$. (This notation is not standard; should you use it on a contest, do not neglect to say so in the opening lines of your solution.)

Here is how it works. First, we consider $\measuredangle ABC$ to be *positive* if the vertices A, B, C appear in clockwise order, and *negative* otherwise. In particular, $\measuredangle ABC \neq \measuredangle CBA$; they are negatives. See Figure 1.5B.

Then, we are taking the angles modulo 180°. For example,

$$-150^\circ = 30^\circ = 210^\circ.$$

Why on earth would we adopt such a strange convention? The key is that our Theorem 1.9 can now be rewritten as follows.

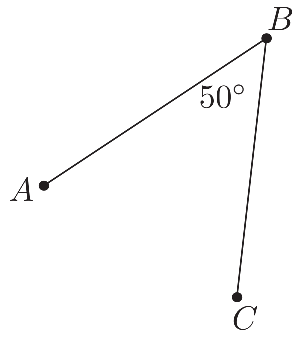

Figure 1.5B. Here, $\measuredangle ABC = 50^\circ$ and $\measuredangle CBA = -50^\circ$.

Theorem 1.22 (Cyclic Quadrilaterals with Directed Angles). *Points A, B, X, Y lie on a circle if and only if*

$$\measuredangle AXB = \measuredangle AYB.$$

This seems too good to be true, as we have dropped the convex condition—there is now only one case of the theorem. In other words, as long as we direct our angles, we no longer have to worry about configuration issues when applying Theorem 1.9.

Problem 1.23. Verify that parts (ii) and (iii) of Theorem 1.9 match the description in Theorem 1.22.

We present some more convenient truths in the following proposition.

Proposition 1.24 (Directed Angles). *For any distinct points A, B, C, P in the plane, we have the following rules.*

Oblivion. $\measuredangle APA = 0$.

Anti-Reflexivity. $\measuredangle ABC = -\measuredangle CBA$.

Replacement. $\measuredangle PBA = \measuredangle PBC$ *if and only if A, B, C are collinear. (What happens when $P = A$?) Equivalently, if C lies on line BA, then the A in $\measuredangle PBA$ may be replaced by C.*

Right Angles. *If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90^\circ$.*

Directed Angle Addition. $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.

Triangle Sum. $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.

Isosceles Triangles. $AB = AC$ *if and only if* $\measuredangle ACB = \measuredangle CBA$.

Inscribed Angle Theorem. *If (ABC) has center P, then $\measuredangle APB = 2\measuredangle ACB$.*

Parallel Lines. *If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.*

One thing we have to be careful about is that $2\measuredangle ABC = 2\measuredangle XYZ$ does not imply $\measuredangle ABC = \measuredangle XYZ$, because we are taking angles modulo 180°. Hence it does not make sense to take half of a directed angle.[†]

Problem 1.25. Convince yourself that all the claims in Proposition 1.24 are correct.

[†] Because of this, it is customary to take arc measures modulo 360°. We may then write the inscribed angle theorem as $\measuredangle ABC = \frac{1}{2}\widehat{AC}$. This is okay since $\measuredangle ABC$ is taken mod 180° but $\widehat{AC}$ is taken mod 360°.

Directed angles are quite counterintuitive at first, but with a little practice they become much more natural. The right way to think about them is to solve the problem for a specific configuration, but write down all statements in terms of directed angles. The solution for a specific configuration then automatically applies to all configurations.

Before moving in to a less trivial example, let us finish the issue with the orthic triangle once and for all.

Example 1.26. Let H be the orthocenter of $\triangle ABC$, acute or not. Using directed angles, show that $AEHF$, $BFHD$, $CDHE$, $BEFC$, $CFDA$, and $ADEB$ are cyclic.

Solution. We know that

$$90^\circ = \measuredangle ADB = \measuredangle ADC$$
$$90^\circ = \measuredangle BEC = \measuredangle BEA$$
$$90^\circ = \measuredangle CFA = \measuredangle CFB$$

because of right angles. Then

$$\measuredangle AEH = \measuredangle AEB = -\measuredangle BEA = -90^\circ = 90^\circ$$

and

$$\measuredangle AFH = \measuredangle AFC = -\measuredangle CFA = -90^\circ = 90^\circ$$

so A, E, F, H are concyclic. Also,

$$\measuredangle BFC = -\measuredangle CFB = -90^\circ = 90^\circ = \measuredangle BEC$$

so B, E, F, C are concyclic. The other quadrilaterals have similar stories. □

We conclude with one final example.

Lemma 1.27 (Miquel Point of a Triangle). *Points D, E, F lie on lines BC, CA, and AB of $\triangle ABC$, respectively. Then there exists a point lying on all three circles (AEF), (BFD), (CDE).*

This point is often called the **Miquel point** of the triangle.

It should be clear by looking at Figure 1.5C that many, many configurations are possible. Trying to handle this with standard angles would be quite messy. Fortunately, we can get them all in one go with directed angles.

Let K be the intersection of (BFD) and (CDE) other than D. The goal is to show that $AFEK$ is cyclic as well. For the case when K is inside $\triangle ABC$, this is an easy angle chase. All we need to do is use the corresponding statements with directed angles for each step.

We strongly encourage readers to try this themselves before reading the solution that follows.

First, here is the solution for the first configuration of Figure 1.5C. Define K as above. Now we just notice that $\angle FKD = 180^\circ - B$ and $\angle EKD = 180^\circ - C$. Consequently, $\angle FKE = 360^\circ - (180^\circ - C) - (180^\circ - B) = B + C = 180^\circ - A$ and $AFEK$ is cyclic. Now we just need to convert this into directed angles.

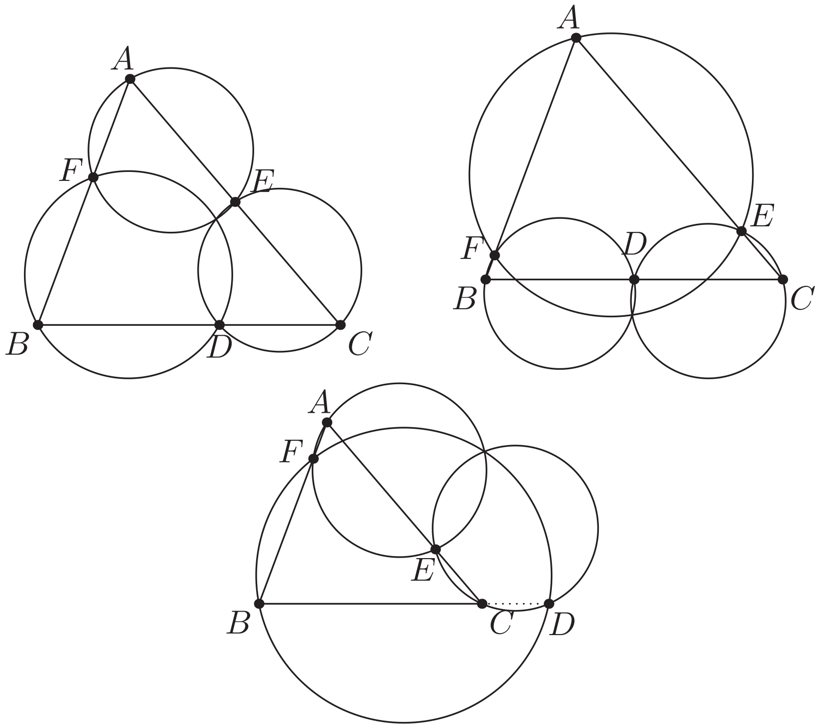

Figure 1.5C. The Miquel point, as in Lemma 1.27.

Proof. The first two claims are just

$$\measuredangle FKD = \measuredangle FBD = \measuredangle ABC \text{ and } \measuredangle DKE = \measuredangle DCE = \measuredangle BCA.$$

We also know that

$$\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0 \text{ and } \measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$$

The first equation represents the fact that the sum of the angles at K is 360°; the second is the fact that the sum of the angles in a triangle is 180°. From here we derive that $\measuredangle CAB = \measuredangle EKF$. But $\measuredangle CAB = \measuredangle EAF$; hence $\measuredangle EAF = \measuredangle EKF$ as desired. □

Having hopefully convinced you that directed angles are natural and often useful, let us provide a warning on when not to use them. Most importantly, you should not use directed angles when the problem only works for a certain configuration! An example of this is Problem 1.38; the problem statement becomes false if the quadrilateral is instead $ABDC$. You should also avoid using directed angles if you need to invoke trigonometry, or if you need to take half an angle (as in Problem 1.38 again). These operations do not make sense modulo 180°.

Problems for this Section

Problem 1.28. We claimed that $\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0$ in the above proof. Verify this using Proposition 1.24.

Problem 1.29. Show that for any distinct points A, B, C, D we have $\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0$. **Hints**: 114 645

Lemma 1.30. *Points A, B, C lie on a circle with center O. Show that $\measuredangle OAC = 90^\circ - \measuredangle CBA$. (This is not completely trivial.)* **Hints**: 8 530 109

1.6 Tangents to Circles and Phantom Points

Here we introduce one final configuration and one general technique.

First, we discuss the **tangents** to a circle. In many ways, one can think of it as Theorem 1.22 applied to the "quadrilateral" $AABC$. Indeed, consider a point X on the circle and the line XA. As we move X closer to A, the line XA approaches the tangent at A. The limiting case becomes the theorem below.

Proposition 1.31 (Tangent Criterion). *Suppose $\triangle ABC$ is inscribed in a circle with center O. Let P be a point in the plane. Then the following are equivalent:*

(i) $\overline{PA}$ *is tangent to* (ABC).
(ii) $\overline{OA} \perp \overline{AP}$.
(iii) $\measuredangle PAB = \measuredangle ACB$.

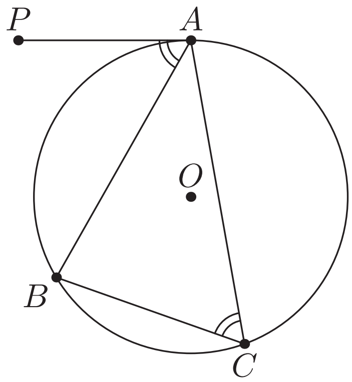

Figure 1.6A. PA is a tangent to (ABC). See Proposition 1.31.

In the following example we also introduce the technique of adding a **phantom point**. (This general theme is sometimes also called **reverse reconstruction**.)

Example 1.32. Let ABC be an acute triangle with circumcenter O, and let K be a point such that $\overline{KA}$ is tangent to (ABC) and $\angle KCB = 90^\circ$. Point D lies on $\overline{BC}$ such that $\overline{KD} \parallel \overline{AB}$. Show that line $\overline{DO}$ passes through A.

This problem is perhaps a bit trickier to solve directly, because we have not developed any tools to show that three points are collinear. (We will!) But here is a different idea. We define a phantom point D' as the intersection of ray AO with $\overline{BC}$. If we can show that $\overline{KD'} \parallel \overline{AB}$, then this will prove $D' = D$, because there is only one point on $\overline{BC}$ with $\overline{KD} \parallel \overline{AB}$.

Fortunately, this can be done with merely the angle chasing that we know earlier. We leave it as Problem 1.33. As a hint, you will have to use both parts of Proposition 1.31.

We have actually encountered a similar idea before, in our proof of Lemma 1.27. The idea was to let (BDF) and (CDE) intersect at a point K, and then show that K was on the

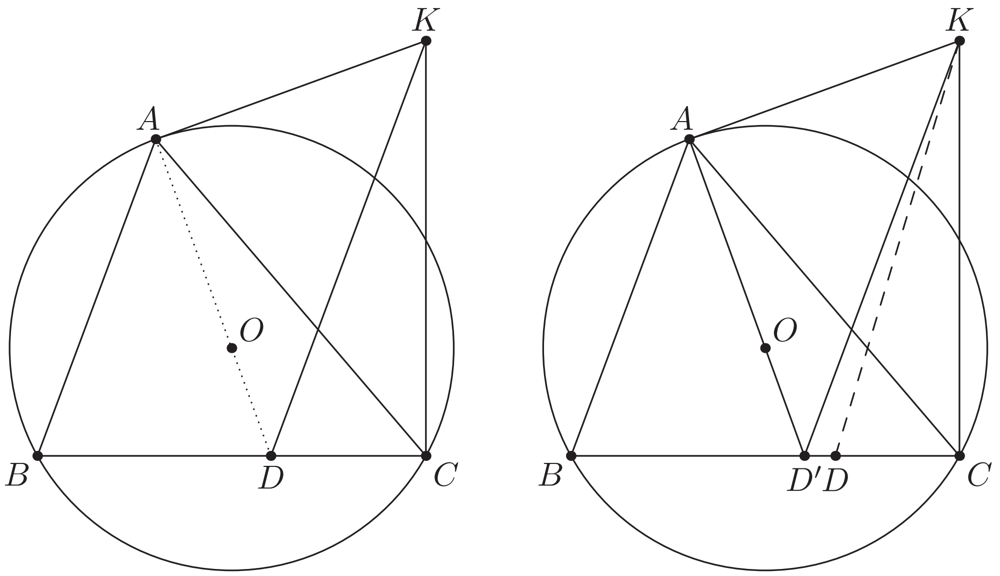

Figure 1.6B. Example 1.32, and the phantom point.

third circle as well. This theme is common in geometry. A second example where phantom points are helpful is Lemma 1.45 on page 19.

It is worth noting that solutions using phantom points can often (but not always) be rearranged to avoid them, although such solutions may be much less natural. For example, another way to solve Example 1.32 is to show that $\measuredangle KAO = \measuredangle KAD$. Problem 1.34 is the most common example of a problem that is not easy to rewrite without phantom points.

Problems for this Section

Problem 1.33. Let ABC be a triangle and let ray AO meet $\overline{BC}$ at D'. Point K is selected so that $\overline{KA}$ is tangent to (ABC) and $\angle KC = 90^\circ$. Prove that $\overline{KD'} \parallel \overline{AB}$.

Problem 1.34. In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Prove that the points A, B, C, K are concyclic.
Hints: 356 101

1.7 Solving a Problem from the IMO Shortlist

To conclude the chapter, we leave the reader with one last example problem. We hope the discussion is instructive.

Example 1.35 (Shortlist 2010/G1). Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

In this problem there are two possible configurations. Directed angles allows us to handle both, but let us focus on just one—say P_2 and Q_2.

The first thing we notice is the orthic triangle. Because of it we should keep the results of Lemma 1.14 close at heart. Additionally, we are essentially given that $ACBP_2$ is a cyclic

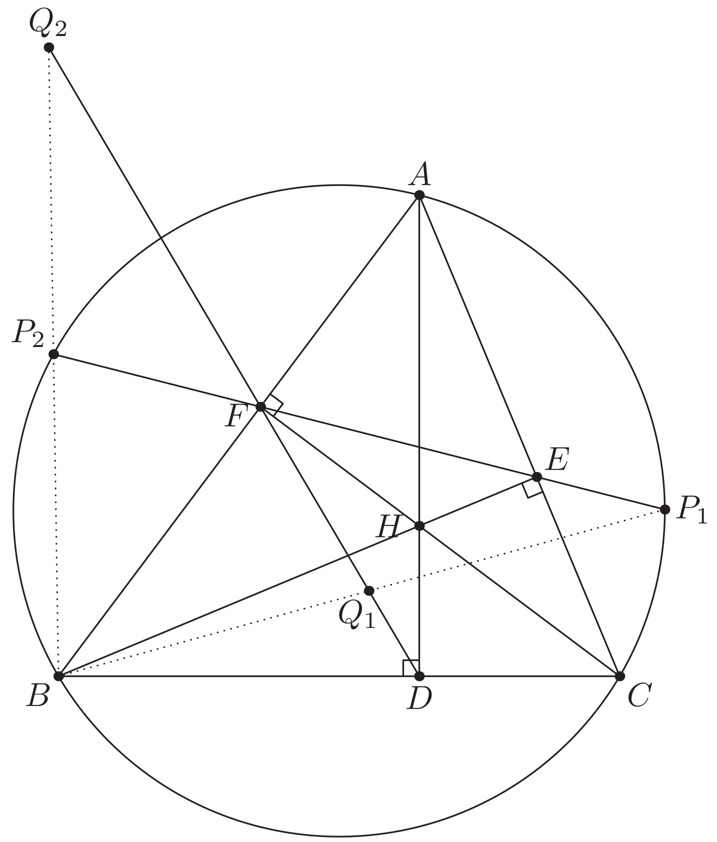

Figure 1.7A. IMO Shortlist 2010, Problem G1 (Example 1.35).

quadrilateral. Let us see what we can do with that. The conclusion $AP_2 = AQ_2$ seems better expressed in terms of angles—we want to show that $\measuredangle AQ_2P_2 = \measuredangle Q_2P_2A$. Now we already know $\measuredangle Q_2P_2A$, because

$$\measuredangle Q_2P_2A = \measuredangle BP_2A = \measuredangle BCA$$

so it is equivalent to compute $\measuredangle AQ_2P_2$.

There are two ways to realize the next step. The first is wishful thinking—the hope that a convenient cyclic quadrilateral will give us $\measuredangle AQ_2P_2$. The second way is to have a scaled diagram at hand. Either way, we stumble upon the following hope: might AQ_2P_2F be cyclic? It certainly looks like it in the diagram.

How might we prove that AQ_2P_2F is cyclic? Trying to use supplementary angles seems not as hopeful, because this is what we want to use as a final step. However, inscribed arcs seems more promising. We already know $\measuredangle AP_2Q_2 = \measuredangle ACB$. Might we be able to find AFQ_2? Yes—we know that

$$\measuredangle AFQ_2 = \measuredangle AFD$$

and now we are certain this will succeed, because $\measuredangle AFD$ is entirely within the realm of $\triangle ABC$ and its orthic triangle. In other words, we have eliminated P and Q. In fact,

$$\measuredangle AFD = \measuredangle ACD = \measuredangle ACB$$

since $AFDC$ is cyclic. This solves the problem for P_2 and Q_2. Because we have been careful to direct all the angles, this automatically solves the case P_1 and Q_1 as well—and this is why directed angles are useful.

It is important to realize that the above is not a well-written proof, but instead a description of how to arrive at the solution. Below is an example of how to write the proof in a contest—one direction only (so without working backwards like we did at first), and without the motivation. Follow along in the following proof with P_1 and Q_1, checking that the directed angles work out.

Solution to Example 1.35. First, because $APBC$ and $AFDC$ are cyclic,

$$\measuredangle QPA = \measuredangle BPA = \measuredangle BCA = \measuredangle DCA = \measuredangle DFA = \measuredangle QFA.$$

Therefore, we see $AFPQ$ is cyclic. Then

$$\measuredangle AQP = \measuredangle AFP = \measuredangle AFE = \measuredangle AHE = \measuredangle DHE = \measuredangle DCE = \measuredangle BCA.$$

We deduce that $\measuredangle AQP = \measuredangle BCA = \measuredangle QPA$ which is enough to imply that $\triangle APQ$ is isosceles with $AP = AQ$. □

This problem is much easier if Lemma 1.14 is kept in mind. In that case, the only key observation is that $AFPQ$ is cyclic. As we saw above, one way to make this key observation is to merely peruse the diagram for quadrilaterals that appear cyclic. That is why it is often a good idea, on any contest problem, to draw a scaled diagram using ruler and compass—in fact, preferably more than one diagram. This often gives away intermediate steps in the problem, prevents you from missing obvious facts, or gives you something to attempt to prove. It will also prevent you from wasting time trying to prove false statements.

1.8 Problems

Problem 1.36. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90^\circ$. Prove that $\overline{AO}$ bisects $\angle BAE$. **Hints**: 18 115 **Sol:** p.241

Problem 1.37 (BAMO 1999/2). Let $O = (0, 0)$, $A = (0, a)$, and $B = (0, b)$, where $0 < a < b$ are reals. Let Γ be a circle with diameter $\overline{AB}$ and let P be any other point on Γ. Line PA meets the x-axis again at Q. Prove that $\angle BQP = \angle BOP$. **Hints**: 635 100

Problem 1.38. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Prove that I_1I_2BC is cyclic. **Hints**: 684 569

Problem 1.39 (CGMO 2012/5). Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Let O denote the circumcenter of $\triangle BCI$.

Prove that $\angle ODB = \angle OEC$. **Hints**: 643 89 **Sol:** p.241

Problem 1.40 (Canada 1991/3). Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle. **Hints**: 455 186 169

Problem 1.41 (Russian Olympiad 1996). Points E and F are on side $\overline{BC}$ of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$. **Hints**: 245 614

Lemma 1.42. *Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc $\widehat{BC}$ not containing A and define Y, Z similarly. Show that the orthocenter of XYZ is the incenter I of ABC.* **Hints**: 432 21 326 195

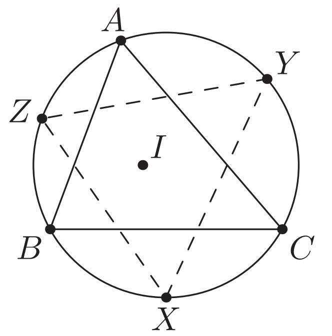

Figure 1.8A. Lemma 1.42. I is the orthocenter of $\triangle XYZ$.

Problem 1.43 (JMO 2011/5). Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$.

Prove that $\overline{BE}$ bisects $\overline{AC}$. **Hints**: 401 575 **Sol:** p.242

Lemma 1.44 (Three Tangents). *Let ABC be an acute triangle. Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$, and denote by M the midpoint of $\overline{BC}$. Prove that $\overline{ME}$, $\overline{MF}$, and the line through A parallel to $\overline{BC}$ are all tangents to (AEF).* **Hints**: 24 335

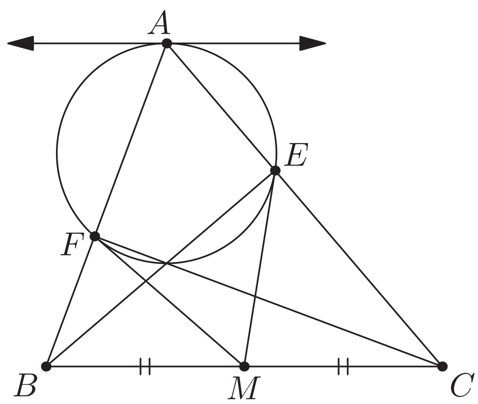

Figure 1.8B. Lemma 1.44, involving tangents to (AEF).

Lemma 1.45 (Right Angles on Incircle Chord). *The incircle of $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Ray BI meets line EF at K. Show that $\overline{BK} \perp \overline{CK}$. Then show K lies on line MN.* **Hints**: 460 84

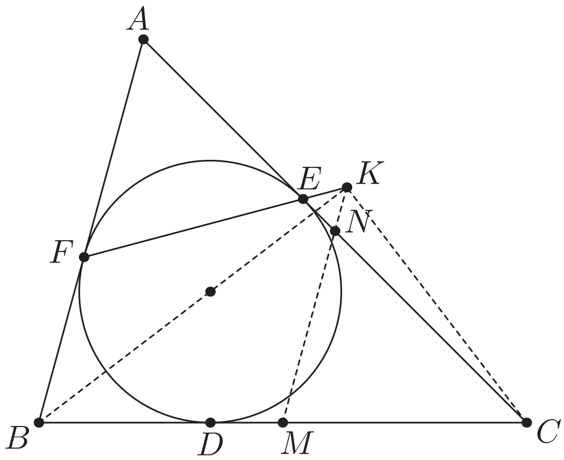

Figure 1.8C. Diagram for Lemma 1.45.

Problem 1.46 (Canada 1997/4). The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$. **Hints**: 386 110 214 **Sol:** p.242

Problem 1.47 (IMO 2006/1). Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$. **Hints**: 212 453 670

Lemma 1.48 (Simson Line). *Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.* **Hints**: 278 502 **Sol:** p.243

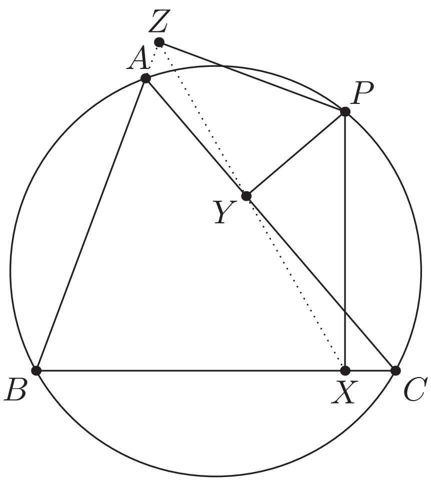

Figure 1.8D. Lemma 1.48; the **Simson line**.

Problem 1.49 (USAMO 2010/1). Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB. **Hint**: 661

Problem 1.50 (IMO 2013/4). Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear. **Hints**: 106 157 15 **Sol:** p.243

Problem 1.51 (IMO 1985/1). A circle has center on the side $\overline{AB}$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. **Hints**: 36 201

CHAPTER 2

Circles

Construct a circle of radius zero...

Although it is often an intermediate step, angle chasing is usually not enough to solve a problem completely. In this chapter, we develop some other fundamental tools involving circles.

2.1 Orientations of Similar Triangles

You probably already know the similarity criterion for triangles. Similar triangles are useful because they let us convert angle information into lengths. This leads to the power of a point theorem, arguably the most common sets of similar triangles.

In preparation for the upcoming section, we develop the notion of similar triangles that are similarly oriented and oppositely oriented.

Here is how it works. Consider triangles ABC and XYZ. We say they are **directly similar**, or similar and **similarly oriented**, if

$$\measuredangle ABC = \measuredangle XYZ, \ \measuredangle BCA = \measuredangle YZX, \ \text{and} \ \measuredangle CAB = \measuredangle ZXY.$$

We say they are **oppositely similar**, or similar and **oppositely oriented**, if

$$\measuredangle ABC = -\measuredangle XYZ, \ \measuredangle BCA = -\measuredangle YZX, \ \text{and} \ \measuredangle CAB = -\measuredangle ZXY.$$

If they are either directly similar or oppositely similar, then they are **similar**. We write $\triangle ABC \sim \triangle XYZ$ in this case. See Figure 2.1A for an illustration.

Two of the angle equalities imply the third, so this is essentially directed AA. Remember to pay attention to the order of the points.

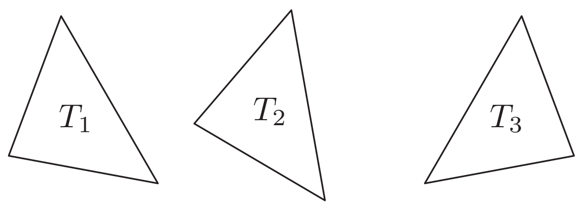

Figure 2.1A. T_1 is directly similar to T_2 and oppositely to T_3.

The upshot of this is that we may continue to use directed angles when proving triangles are similar; we just need to be a little more careful. In any case, as you probably already know, similar triangles also produce ratios of lengths.

Proposition 2.1 (Similar Triangles). *The following are equivalent for triangles ABC and XYZ.*

(i) $\triangle ABC \sim \triangle XYZ$.
(ii) (AA) $\angle A = \angle X$ *and* $\angle B = \angle Y$.
(iii) (SAS) $\angle B = \angle Y$, *and* $AB : XY = BC : YZ$.
(iv) (SSS) $AB : XY = BC : YZ = CA : ZX$.

Thus, lengths (particularly their ratios) can induce similar triangles and vice versa. It is important to notice that SAS similarity does not have a directed form; see Problem 2.2. In the context of angle chasing, we are interested in showing that two triangles are similar using directed AA, and then using the resulting length information to finish the problem. The power of a point theorem in the next section is perhaps the greatest demonstration. However, we remind the reader that angle chasing is only a small part of olympiad geometry, and not to overuse it.

Problem for this Section

Problem 2.2. Find an example of two triangles ABC and XYZ such that $AB : XY = BC : YZ$, $\measuredangle BCA = \measuredangle YZX$, but $\triangle ABC$ and $\triangle XYZ$ are not similar.

2.2 Power of a Point

Cyclic quadrilaterals have many equal angles, so it should come as no surprise that we should be able to find some similar triangles. Let us see what length relations we can deduce.

Consider four points A, B, X, Y lying on a circle. Let line AB and line XY intersect at P. See Figure 2.2A.

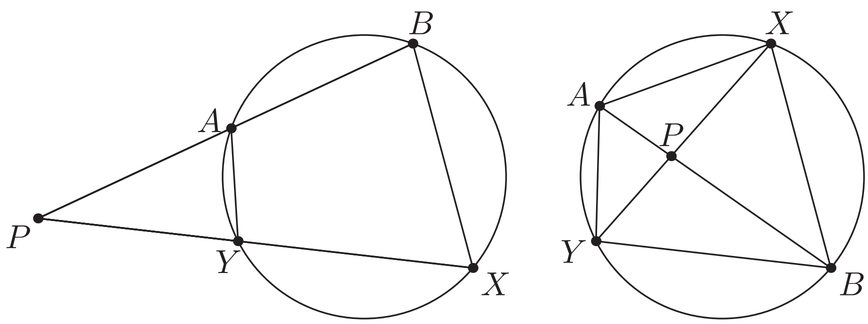

Figure 2.2A. Configurations in power of a point.

A simple directed angle chase gives that

$$\measuredangle PAY = \measuredangle BAY = \measuredangle BXY = \measuredangle BXP = -\measuredangle PXB$$

and

$$\measuredangle AYP = \measuredangle AYX = \measuredangle ABX = \measuredangle PBX = -\measuredangle XBP.$$

As a result, we deduce that $\triangle PAY$ is oppositely similar to $\triangle PXB$.

Therefore, we derive

$$\frac{PA}{PY} = \frac{PX}{PB}$$

or

$$PA \cdot PB = PX \cdot PY.$$

This is the heart of the theorem. Another way to think of this is that the quantity $PA \cdot PB$ does not depend on the choice of line AB, but instead only on the point P. In particular, if we choose line AB to pass through the center of the circle, we obtain that $PA \cdot PB = |PO - r||PO + r|$ where O and r are the center and radius of ω, respectively. In light of this, we define the **power of P** with respect to the circle ω by

$$\text{Pow}_\omega(P) = OP^2 - r^2.$$

This quantity may be negative. Actually, the sign allows us to detect whether P lies inside the circle or not. With this definition we obtain the following properties.

Theorem 2.3 (Power of a Point). *Consider a circle ω and an arbitrary point P.*

(a) The quantity $\text{Pow}_\omega(P)$ is positive, zero, or negative according to whether P is outside, on, or inside ω, respectively.
(b) If ℓ is a line through P intersecting ω at two distinct points X and Y, then

$$PX \cdot PY = |\text{Pow}_\omega(P)|.$$

(c) If P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then

$$PA^2 = \text{Pow}_\omega(P).$$

Perhaps even more important is the converse of the power of a point, which allows us to find cyclic quadrilaterals based on length. Here it is.

Theorem 2.4 (Converse of the Power of a Point). *Let A, B, X, Y be four distinct points in the plane and let lines AB and XY intersect at P. Suppose that either P lies in both of the segments $\overline{AB}$ and $\overline{XY}$, or in neither segment. If $PA \cdot PB = PX \cdot PY$, then A, B, X, Y are concyclic.*

Proof. The proof is by phantom points (see Example 1.32, say). Let line XP meet (ABX) at Y'. Then A, B, X, Y' are concyclic. Therefore, by power of a point, $PA \cdot PB = PX \cdot PY'$. Yet we are given $PA \cdot PB = PX \cdot PY$. This implies $PY = PY'$.

We are not quite done! We would like that $Y = Y'$, but $PY = PY'$ is not quite enough. See Figure 2.2B. It is possible that Y and Y' are reflections across point P.

Fortunately, the final condition now comes in. Assume for the sake of contradiction that $Y \neq Y'$; then Y and Y' are reflections across P. The fact that A, B, X, Y' are concyclic implies that P lies in both or neither of $\overline{AB}$ and $\overline{XY'}$. Either way, this changes if we consider $\overline{AB}$ and $\overline{XY}$. This violates the second hypothesis of the theorem, contradiction. $\square$

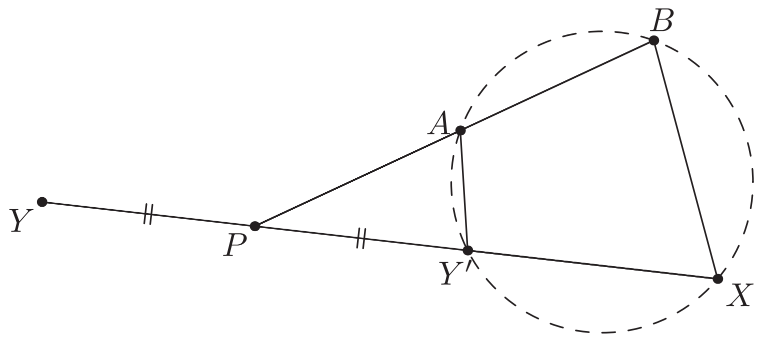

Figure 2.2B. It's a trap! $PA \cdot PB = PX \cdot PY$ almost implies concyclic, but not quite.

As you might guess, the above theorem often provides a bridge between angle chasing and lengths. In fact, it can appear in even more unexpected ways. See the next section.

Problems for this Section

Problem 2.5. Prove Theorem 2.3.

Problem 2.6. Let ABC be a right triangle with $\angle ACB = 90^\circ$. Give a proof of the Pythagorean theorem using Figure 2.2C. (Make sure to avoid a circular proof.)

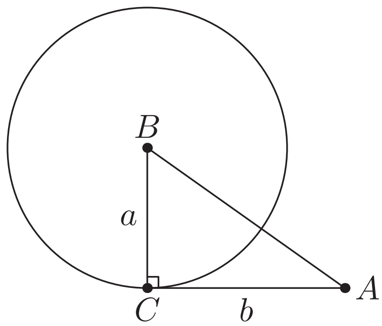

Figure 2.2C. A proof of the Pythagorean theorem.

2.3 The Radical Axis and Radical Center

We start this section with a teaser.

Example 2.7. Three circles intersect as in Figure 2.3A. Prove that the common chords are concurrent.

This seems totally beyond the reach of angle chasing, and indeed it is. The key to unlocking this is the radical axis.

Given two circles ω_1 and ω_2 with distinct centers, the **radical axis** of the circles is the set of points P such that

$$\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P).$$

At first, this seems completely arbitrary. What could possibly be interesting about having equal power to two circles? Surprisingly, the situation is almost the opposite.

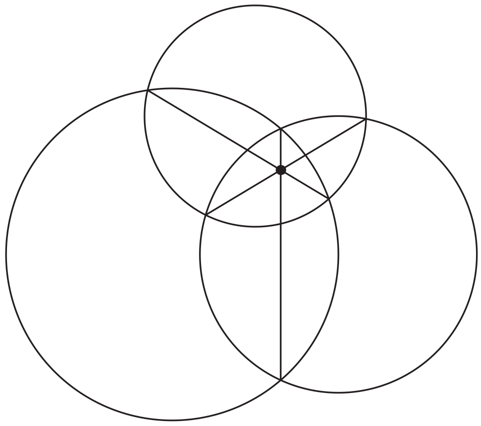

Figure 2.3A. The common chords are concurrent.

Theorem 2.8 (Radical Axis). *Let ω_1 and ω_2 be circles with distinct centers O_1 and O_2. The radical axis of ω_1 and ω_2 is a straight line perpendicular to $\overline{O_1O_2}$.*

In particular, if ω_1 and ω_2 intersect at two points A and B, then the radical axis is line AB.

An illustration is in Figure 2.3B.

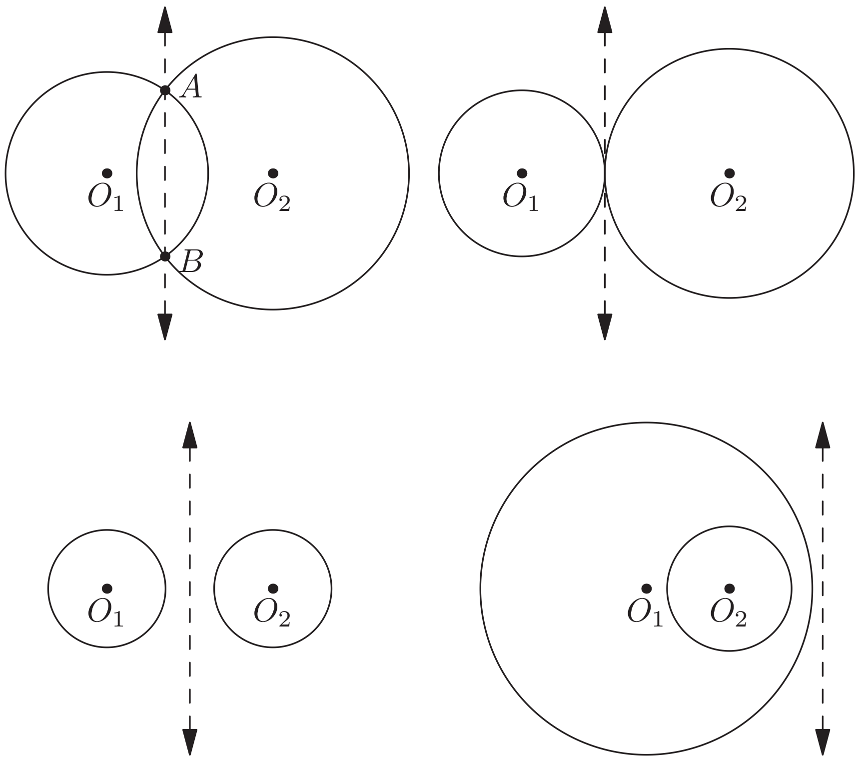

Figure 2.3B. Radical axes on display.

Proof. This is one of the nicer applications of Cartesian coordinates—we are motivated to do so by the squares of lengths appearing, and the perpendicularity of the lines. Suppose that $O_1 = (a, 0)$ and $O_2 = (b, 0)$ in the coordinate plane and the circles have radii r_1 and r_2 respectively. Then for any point $P = (x, y)$ we have

$$\operatorname{Pow}_{\omega_1}(P) = O_1P^2 - r_1^2 = (x-a)^2 + y^2 - r_1^2.$$

Similarly,

$$\text{Pow}_{\omega_2}(P) = O_2P^2 - r_2^2 = (x-b)^2 + y^2 - r_2^2.$$

Equating the two, we find the radical axis of ω_1 and ω_2 is the set of points $P = (x, y)$ satisfying

$$\begin{aligned} 0 &= \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P) \\ &= \left[(x-a)^2 + y^2 - r_1^2\right] - \left[(x-b)^2 + y^2 - r_2^2\right] \\ &= (-2a + 2b)x + \left(a^2 - b^2 + r_2^2 - r_1^2\right) \end{aligned}$$

which is a straight line perpendicular to the x-axis (as $-2a + 2b \neq 0$). This implies the result.

The second part is an immediately corollary. The points A and B have equal power (namely zero) to both circles; therefore, both A and B lie on the radical axis. Consequently, the radical axis must be the line AB itself. □

As a side remark, you might have realized in the proof that the standard equation of a circle $(x-m)^2 + (y-n)^2 - r^2 = 0$ is actually just the expansion of $\text{Pow}_\omega((x, y)) = 0$. That is, the expression $(x-m)^2 + (y-n)^2 - r^2$ actually yields the power of the point (x, y) in Cartesian coordinates to the circle centered at (m, n) with radius r.

The power of Theorem 2.8 (no pun intended) is the fact that it is essentially an "if and only if" statement. That is, a point has equal power to both circles if and only if it lies on the radical axis, which we know much about.

Let us now return to the problem we saw at the beginning of this section. Some of you may already be able to guess the ending.

Proof of Example 2.7. The common chords are radical axes. Let ℓ_{12} be the radical axis of ω_1 and ω_2, and let ℓ_{23} be the radical axis of ω_2 and ω_3.

Let P be the intersection of these two lines. Then

$$P \in \ell_{12} \Rightarrow \text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P)$$

and

$$P \in \ell_{23} \Rightarrow \text{Pow}_{\omega_2}(P) = \text{Pow}_{\omega_3}(P)$$

which implies $\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_3}(P)$. Hence $P \in \ell_{31}$ and accordingly we discover that all three lines pass through P. □

In general, consider three circles with distinct centers O_1, O_2, O_3. In light of the discussion above, there are two possibilities.

1. Usually, the pairwise radical axes concur at a single point K. In that case, we call K the **radical center** of the three circles.
2. Occasionally, the three radical axes will be pairwise parallel (or even the same line). Because the radical axis of two circles is perpendicular to the line joining its centers, this (annoying) case can only occur if O_1, O_2, O_3 are collinear.

It is easy to see that these are the only possibilities; whenever two radical axes intersect, then the third one must pass through their intersection point.

We should also recognize that the converse to Example 2.7 is also true. Here is the full configuration.

Theorem 2.9 (Radical Center of Intersecting Circles). *Let ω_1 and ω_2 be two circles with centers O_1 and O_2. Select points A and B on ω_1 and points C and D on ω_2. Then the following are equivalent:*

(a) A, B, C, D lie on a circle with center O_3 not on line O_1O_2.
(b) Lines AB and CD intersect on the radical axis of ω_1 and ω_2.

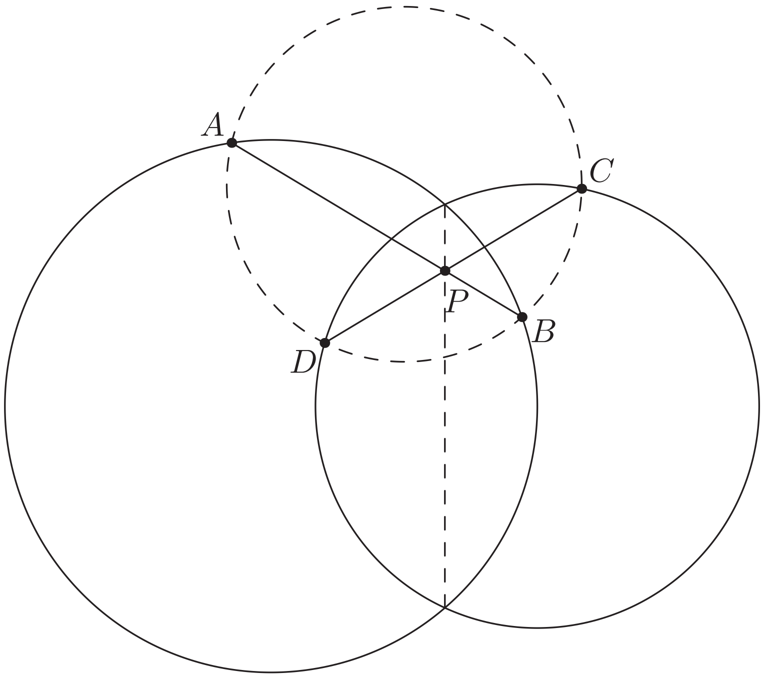

Figure 2.3C. The converse is also true. See Theorem 2.9.

Proof. We have already shown one direction. Now suppose lines AB and CD intersect at P, and that P lies on the radical axis. Then

$$\pm PA \cdot PB = \operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P) = \pm PC \cdot PD.$$

We need one final remark: we see that $\operatorname{Pow}_{\omega_1}(P) > 0$ if and only if P lies strictly between A and B. Similarly, $\operatorname{Pow}_{\omega_2}(P) > 0$ if and only if P lies strictly between C and D. Because $\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P)$, we have the good case of Theorem 2.4. Hence, because $PA \cdot PB = PC \cdot PD$, we conclude that A, B, C, D are concyclic. Because lines AB and CD are not parallel, it must also be the case that the points O_1, O_2, O_3 are not collinear. $\square$

We have been very careful in our examples above to check that the power of a point holds in the right direction, and to treat the two cases "concurrent" or "all parallel". In practice, this is more rarely an issue, because the specific configuration in an olympiad problem often excludes such pathological configurations. Perhaps one notable exception is USAMO 2009/1 (Example 2.21).

To conclude this section, here is one interesting application of the radical axis that is too surprising to be excluded.

Proposition 2.10. *In a triangle ABC, the circumcenter exists. That is, there is a point O such that $OA = OB = OC$.*

Proof. Construct a circle of radius zero (!) centered at A, and denote it by ω_A. Define ω_B and ω_C similarly. Because the centers are not collinear, we can find their radical center O.

Now we know the powers from O to each of ω_A, ω_B, ω_C are equal. Rephrased, the (squared) length of the "tangents" to each circle are equal: that is, $OA^2 = OB^2 = OC^2$. (To see that OA^2 really is the power, just use $\text{Pow}_{\omega_A}(O) = OA^2 - 0^2 = OA^2$.) From here we derive that $OA = OB = OC$, as required. □

Of course, the radical axes are actually just the perpendicular bisectors of the sides. But this presentation was simply too surprising to forgo. This may be the first time you have seen a circle of radius zero; it will not be the last.

Problems for this Section

Lemma 2.11. *Let ABC be a triangle and consider a point P in its interior. Suppose that $\overline{BC}$ is tangent to the circumcircles of triangles ABP and ACP. Prove that ray AP bisects $\overline{BC}$.*

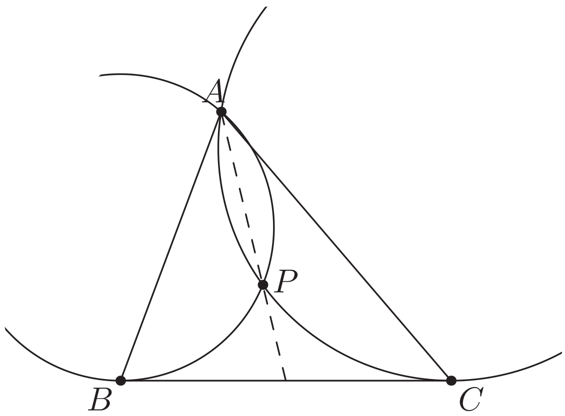

Figure 2.3D. Diagram for Lemma 2.11.

Problem 2.12. Show that the orthocenter of a triangle exists using radical axes. That is, if $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of a triangle ABC, show that the altitudes are concurrent.
Hint: 367

2.4 Coaxial Circles

If a set of circles have the same radical axes, then we say they are **coaxial**. A collection of such circles is called a **pencil** of coaxial circles. In particular, if circles are coaxal, their centers are collinear. (The converse is not true.)

Coaxial circles can arise naturally in the following way.

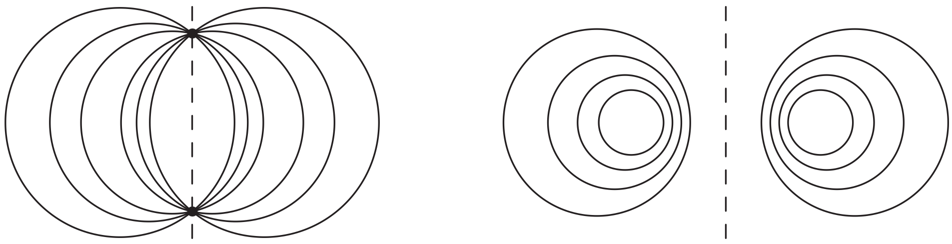

Figure 2.4A. Two pencils of coaxial circles.

Lemma 2.13 (Finding Coaxial Circles). *Three distinct circles Ω_1, Ω_2, Ω_3 pass through a point X. Then their centers are collinear if and only if they share a second common point.*

Proof. Both conditions are equivalent to being coaxial. □

2.5 Revisiting Tangents: The Incenter

We consider again an angle bisector. See Figure 2.5A.

For any point P on the angle bisector, the distances from P to the sides are equal. Consequently, we can draw a circle centered at P tangent to the two sides. Conversely, the two tangents to any circle always have equal length, and the center of that circle lies on the corresponding angle bisector.

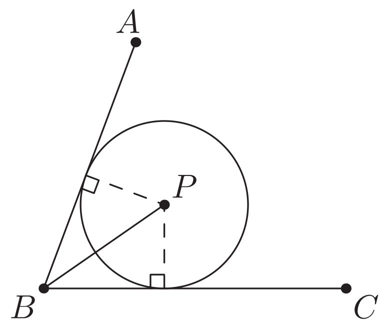

Figure 2.5A. Two tangents to a circle.

From these remarks we can better understand the incenter.

Proposition 2.14. *In any triangle ABC, the angle bisectors concur at a point I, which is the center of a circle inscribed in the triangle.*

Proof. Essentially we are going to complete Figure 2.5A to obtain Figure 2.5B. Let the angle bisectors of $\angle B$ and $\angle C$ intersect at a point I. We claim that I is the desired incenter.

Let D, E, F be the projections of I onto $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Because I is on the angle bisector of $\angle B$, we know that $IF = ID$. Because I is on the angle bisector of $\angle C$, we know that $ID = IE$. (If this reminds you of the proof of the radical center, it should!) Therefore, $IE = IF$, and we deduce that I is also on the angle bisector of $\angle A$. Finally, the circle centered at I with radius $ID = IE = IF$ is evidently tangent to all sides. □

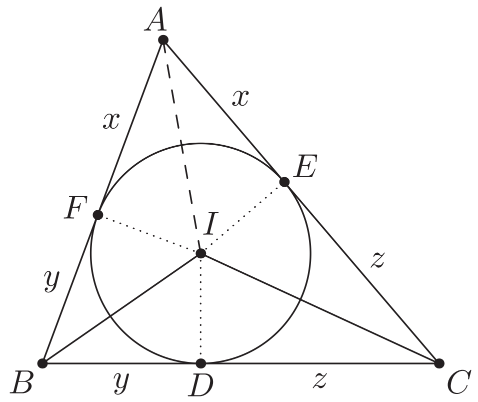

Figure 2.5B. Describing the incircle of a triangle.

The triangle DEF is called the **contact triangle** of $\triangle ABC$.

We can say even more. In Figure 2.5B we have marked the equal lengths induced by the tangents as x, y, and z. Considering each of the sides, this gives us a system of equations of three variables

$$y + z = a$$
$$z + x = b$$
$$x + y = c.$$

Now we can solve for x, y, and z in terms of a, b, c. This is left as an exercise, but we state the result here. (Here $s = \frac{1}{2}(a + b + c)$.)

Lemma 2.15 (Tangents to the Incircle). *If DEF is the contact triangle of $\triangle ABC$, then $AE = AF = s - a$. Similarly, $BF = BD = s - b$ and $CD = CE = s - c$.*

Problem for this Section

Problem 2.16. Prove Lemma 2.15.

2.6 The Excircles

In Lemma 1.18 we briefly alluded the excenter of a triangle. Let us consider it more completely here. The ***A*-excircle** of a triangle ABC is the circle that is tangent to $\overline{BC}$, the extension of $\overline{AB}$ past B, and the extension of $\overline{AC}$ past C. See Figure 2.6A. The ***A*-excenter**, usually denoted I_A, is the center of the A-excircle. The B-excircle and C-excircles are defined similarly and their centers are unsurprisingly called the B-excenter and the C-excenter.

We have to actually check that the A-excircle exists, as it is not entirely obvious from the definition. The proof is exactly analogous to that for the incenter, except with the angle bisector from B replaced with an **external angle bisector**, and similarly for C. As a simple corollary, the incenter of ABC lies on $\overline{AI_A}$.

Now let us see if we can find similar length relations as in the incircle. Let X be the tangency point of the A-excircle on $\overline{BC}$ and B_1 and C_1 the tangency points to rays AB and

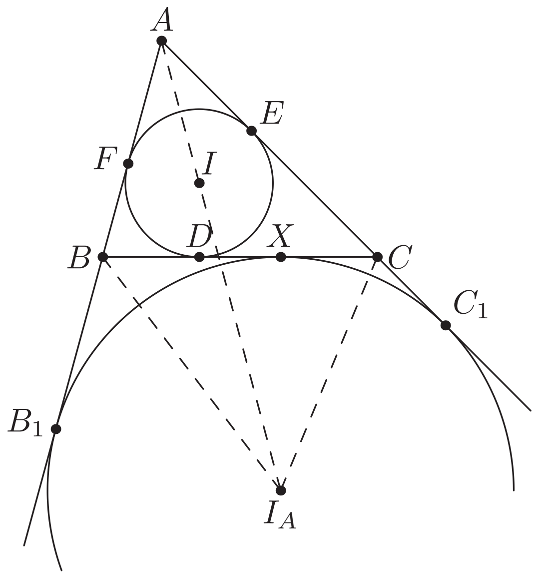

Figure 2.6A. The incircle and A-excircle.

AC. We know that $AB_1 = AC_1$ and that

$$\begin{aligned} AB_1 + AC_1 &= (AB + BB_1) + (AC + CC_1) \\ &= (AB + BX) + (AC + CX) \\ &= AB + AC + BC \\ &= 2s. \end{aligned}$$

We have now obtained the following.

Lemma 2.17 (Tangents to the Excircle). *If AB_1 and AC_1 are the tangents to the A-excircle, then $AB_1 = AC_1 = s$.*

Let us make one last remark: in Figure 2.6A, the triangles AIF and AI_AB_1 are directly similar. (Why?) This lets us relate the ***A*-exradius**, or the radius of the excircle, to the other lengths in the triangle. This exradius is usually denoted r_a. See Lemma 2.19.

Problems for this Section

Problem 2.18. Let the external angle bisectors of B and C in a triangle ABC intersect at I_A. Show that I_A is the center of a circle tangent to $\overline{BC}$, the extension of $\overline{AB}$ through B, and the extension of $\overline{AC}$ through C. Furthermore, show that I_A lies on ray AI.

Lemma 2.19 (Length of Exradius). *Prove that the A-exradius has length*

$$r_a = \frac{s}{s-a} r.$$

Hint: 302

Lemma 2.20. *Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to $\overline{BC}$ at X and D, respectively. Show that $BX = CD$ and $BD = CX$.*

2.7 Example Problems

We finish this chapter with several problems, which we feel are either instructive, classical, or too surprising to not be shared.

Example 2.21 (USAMO 2009/1). Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

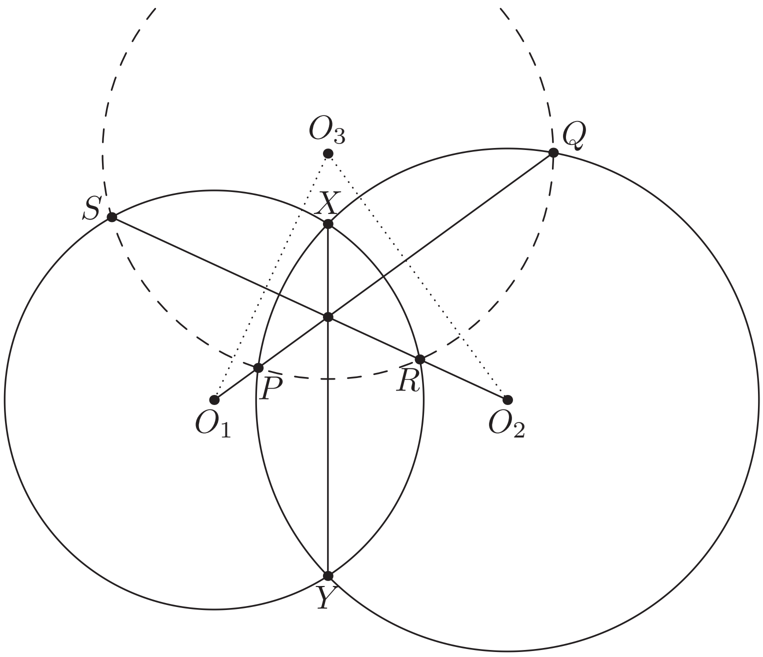

Figure 2.7A. The first problem of the 2009 USAMO.

This was actually a very nasty USAMO problem, in the sense that it was easy to lose partial credit. We will see why.

Let O_3 and ω_3 be the circumcenter and circumcircle, respectively, of the cyclic quadrilateral $PQRS$. After drawing the diagram, we are immediately reminded of our radical axes. In fact, we already know that that lines PQ, RS, and XY concur at a point X, by Theorem 2.9. Call this point H.

Now, what else do we know? Well, glancing at the diagram* it appears that $\overline{O_1O_3} \perp \overline{RS}$. And of course this we know is true, because $\overline{RS}$ is the radical axis of ω_1 an ω_3. Similarly, we notice that $\overline{PQ}$ is perpendicular to O_1O_3.

Focus on $\triangle O_1O_2O_3$. We see that H is its orthocenter. Therefore the altitude from O_3 to $\overline{O_1O_2}$ must pass through H. But line XY is precisely that altitude: it passes through H and is perpendicular to $\overline{O_1O_2}$. Hence, O_3 lies on line XY, and we are done.

Or are we?

Look at Theorem 2.9 again. In order to apply it, we need to know that O_1, O_2, O_3 are not collinear. Unfortunately, this is not always true—see Figure 2.7B.

Fortunately, noticing this case is much harder than actually doing it. We use phantom points. Let O be the midpoint of $\overline{XY}$. (We pick this point because we know this is where O_3

* And you are drawing large scaled diagrams, right?

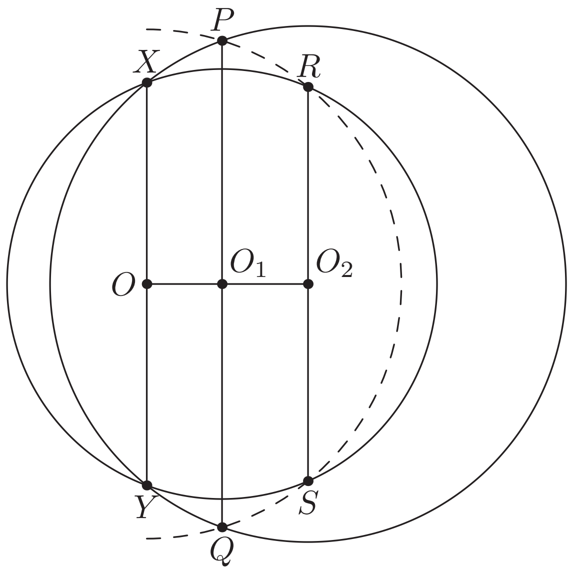

Figure 2.7B. An unnoticed special case.

must be for the problem to hold.) Now we just need to show that $OP = OQ = OR = OS$, from which it will follow that $O = O_3$.

This looks much easier. It should seem like we should be able to compute everything using just repeated applications of the Pythagorean theorem (and the definition of a circle). Trying this,

$$\begin{aligned} OP^2 &= OO_1^2 + O_1P^2 \\ &= OO_1^2 + (O_2P^2 - O_1O_2^2) \\ &= OO_1^2 + r_2^2 - O_1O_2^2. \end{aligned}$$

Now the point P is gone from the expression, but the r_2 needs to go if we hope to get a symmetric expression. We can get rid of it by using $O_2X = r_2 = \sqrt{XO^2 + OO_2^2}$.

$$\begin{aligned} OP^2 &= OO_1^2 + (O_2X^2 + OX^2) - O_1O_2^2 \\ &= OX^2 + OO_1^2 + OO_2^2 - O_1O_2^2 \\ &= \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2. \end{aligned}$$

This is symmetric; the exact same calculations with Q, R, and S yield the same results. We conclude $OP^2 = OQ^2 = OR^2 = OS^2 = \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2$ as desired.

Having presented the perhaps more natural solution above, here is a solution with a more analytic flavor. It carefully avoids the configuration issues in the first solution.

Solution to Example 2.21. Let r_1, r_2, r_3 denote the circumradii of ω_1, ω_2, and ω_3, respectively.

We wish to show that O_3 lies on the radical axis of ω_1 and ω_2. Let us encode the conditions using power of a point. Because O_1 is on the radical axis of ω_2 and ω_3,

$$\begin{aligned}\text{Pow}_{\omega_2}(O_1) &= \text{Pow}_{\omega_3}(O_1)\\ \Rightarrow O_1O_2^2 - r_2^2 &= O_1O_3^2 - r_3^2.\end{aligned}$$

Similarly, because O_2 is on the radical axis of ω_1 and ω_3, we have

$$\begin{aligned}\text{Pow}_{\omega_1}(O_2) &= \text{Pow}_{\omega_3}(O_2)\\ \Rightarrow O_1O_2^2 - r_1^2 &= O_2O_3^2 - r_3^2.\end{aligned}$$

Subtracting the two gives

$$\begin{aligned}(O_1O_2^2 - r_2^2) - (O_1O_2^2 - r_1^2) &= (O_1O_3^2 - r_3^2) - (O_2O_3^2 - r_3^2)\\ \Rightarrow r_1^2 - r_2^2 &= O_1O_3^2 - O_2O_3^2\\ \Rightarrow O_2O_3^2 - r_2^2 &= O_1O_3^2 - r_1^2\\ \Rightarrow \text{Pow}_{\omega_2}(O_3) &= \text{Pow}_{\omega_1}(O_3)\end{aligned}$$

as desired. □

The main idea of this solution is to encode everything in terms of lengths using the radical axis. Effectively, we write down the givens as equations. We also write the desired conclusion as an equation, namely $\text{Pow}_{\omega_2}(O_3) = \text{Pow}_{\omega_1}(O_3)$, then forget about geometry and do algebra. It is an unfortunate irony of olympiad geometry that analytic solutions are often immune to configuration issues that would otherwise plague traditional solutions.

The next example is a classical result of Euler.

Lemma 2.22 (Euler's Theorem). *Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R - 2r)$. In particular, $R \geq 2r$.*

The first thing we notice is that the relation is equivalent to proving $R^2 - OI^2 = 2Rr$. This is power of a point, clear as day. So, we let ray AI hit the circumcircle again at L. Evidently we just need to show

$$AI \cdot IL = 2Rr.$$

This looks much nicer to work with—noticing the power expressions gave us a way to clean up the problem statement, and gives us some structure to work on.

We work backwards for a little bit. The final condition appears like similar triangles. So perhaps we may rewrite it as

$$\frac{AI}{r} = \frac{2R}{IL}.$$

There are not too many ways the left-hand side can show up like that. We drop the altitude from I to $\overline{AB}$ as F. Then $\triangle AIF$ has the ratios that we want. (You can also drop the foot to

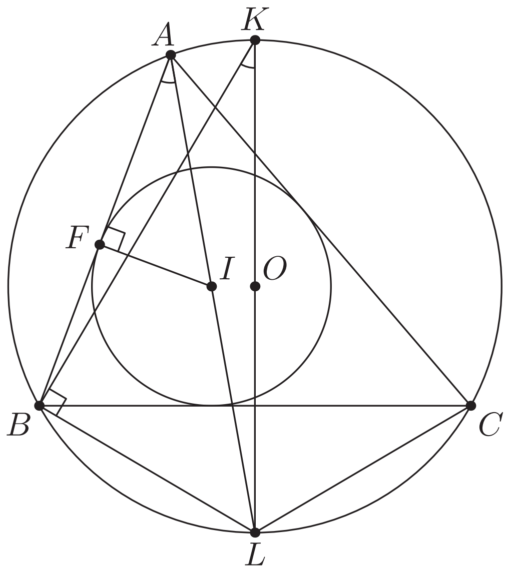

Figure 2.7C. Proving Euler's theorem.

$\overline{AC}$, but this is the same thing.) All that remains is to construct a similar triangle with the lengths $2R$ and IL. Unfortunately, $\overline{IL}$ does not play well in this diagram.

But we hope that by now you recognize $\overline{IL}$ from Lemma 1.18! Write $BL = IL$. Then let K be the point such that $\overline{KL}$ is a diameter of the circle. Then $\triangle KBL$ has the dimensions we want. Could the triangles in question be similar? Yes: $\angle KBL$ and $\angle AFI$ are both right angles, and $\angle BAL = \angle BKL$ by cyclic quadrilaterals. Hence this produces $AI \cdot IL = 2Rr$ and we are done.

As usual, this is not how a solution should be written up in a contest. Solutions should be only written forwards, and without explaining where the steps come from.

Solution to Lemma 2.22. Let ray AI meet the circumcircle again at L and let K be the point diametrically opposite L. Let F be the foot from I to $\overline{AB}$. Notice that $\angle FAI = \angle BAL = \angle BKL$ and $\angle AFI = \angle KBL = 90^\circ$, so

$$\frac{AI}{r} = \frac{AI}{IF} = \frac{KL}{LB} = \frac{2R}{LI}$$

and hence $AI \cdot IL = 2Rr$. Because I lies inside $\triangle ABC$, we deduce the power of I with respect to (ABC) is $2Rr = R^2 - OI^2$. Consequently, $OI^2 = R(R - 2r)$. □

The construction of the diameter appears again in Chapter 3, when we derive the extended law of sines, Theorem 3.1.

Our last example is from the All-Russian Mathematical Olympiad, whose solution is totally unexpected. Please ponder it before reading the solution.

Example 2.23 (Russian Olympiad 2010). Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

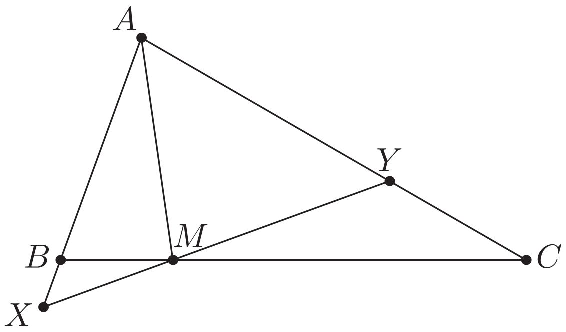

Figure 2.7D. A problem from the All-Russian MO 2010.

What strange conditions have been given. We are told the lengths $AX = AY = 1$ and the perimeter of $\triangle ABC$ is 4, and effectively nothing else. The conclusion, which is an either-or statement, is equally puzzling.

Let us reflect the point A over both X and Y to two points U and V so that $AU = AV = 2$. This seems slightly better, because $AU = AV = 2$ now, and the "two" in the perimeter is now present. But what do we do? Recalling that $s = 2$ in the triangle, we find that U and V are the tangency points of the excircle, call it Γ_a. Set I_A the excenter, tangent to $\overline{BC}$ at T. See Figure 2.7E.

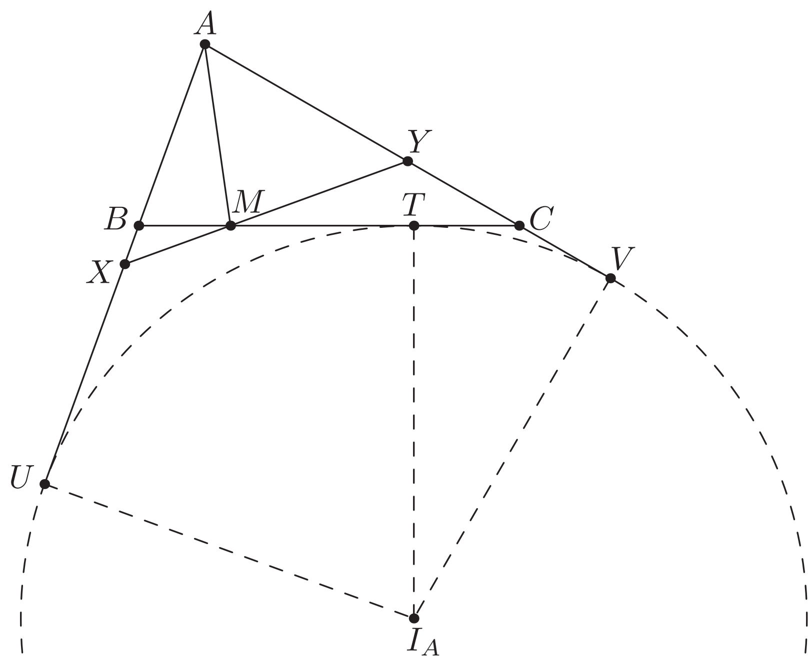

Figure 2.7E. Adding an excircle to handle the conditions.

Looking back, we have now encoded the $AX = AY = 1$ condition as follows: X and Y are the midpoints of the tangents to the A-excircle. We need to show that one of $\triangle ABM$ or $\triangle ACM$ has perimeter equal to the length of the tangent.

Now the question is: how do we use this?

Let us look carefully again at the diagram. It would seem to suggest that in this case, $\triangle ABM$ is the one with perimeter two (and not $\triangle ACM$). What would have to be true in order to obtain the relation $AB + BM + MA = AU$? Trying to bring the lengths closer to the triangle in question, we write $AU = AB + BU = AB + BT$. So we would need $BM + MA = BT$, or $MA = MT$.

So it would appear that the points X, M, Y have the property that their distance to A equals the length of their tangents to the A-excircle. This motivates a last addition to our diagram: construct a circle of radius zero at A, say ω_0. Then X and Y lie on the radical axis of ω_0 and Γ_a; hence so does M! Now we have $MA = MT$, as required.

Now how does the either-or condition come in? Now it is clear: it reflects whether T lies on $\overline{BM}$ or $\overline{CM}$. (It must lie in at least one, because we are told that M lies inside the segment $\overline{BC}$, and the tangency points of the A-excircle to $\overline{BC}$ always lie in this segment as well.) This completes the solution, which we present concisely below.

Solution to Example 2.23. Let I_A be the center of the A-excircle, tangent to $\overline{BC}$ at T, and to the extensions of $\overline{AB}$ and $\overline{AC}$ at U and V. We see that $AU = AV = s = 2$. Then $\overline{XY}$ is the radical axis of the A-excircle and the circle of radius zero at A. Therefore $AM = MT$.

Assume without loss of generality that T lies on $\overline{MC}$, as opposed to $\overline{MB}$. Then $AB + BM + MA = AB + BM + MT = AB + BT = AB + BU = AU = 2$ as desired. □

While we have tried our best to present the solution in a natural way, it is no secret that this is a hard problem by any standard. It is fortunate that such pernicious problems are rare.

2.8 Problems

Lemma 2.24. *Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.* **Hints**: 564 103

Theorem 2.25 (The Pitot Theorem). *Let $ABCD$ be a quadrilateral. If a circle can be inscribed*[†] *in it, prove that $AB + CD = BC + DA$.* **Hint**: 467

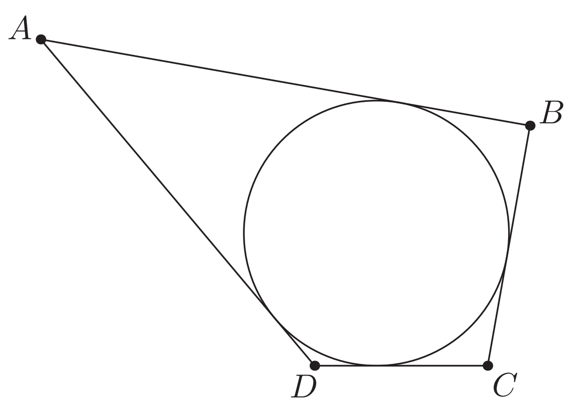

Figure 2.8A. The Pitot theorem: $AB + CD = BC + DA$.

[†] The converse of the Pitot theorem is in fact also true: if $AB + CD = BC + DA$, then a circle can be inscribed inside $ABCD$. Thus, if you ever need to prove $AB + CD = BC + DA$, you may safely replace this with the "inscribed" condition.

Problem 2.26 (USAMO 1990/5). An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle. **Hints**: 260 73 409 **Sol:** p.244

Problem 2.27 (BAMO 2012/4). Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N. **Hints**: 57 77
(b) Prove that no matter where one chooses the point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane. **Hints**: 230 654

Problem 2.28 (JMO 2012/1). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic. **Hints**: 435 601 537 122

Problem 2.29 (IMO 2008/1). Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic. **Hints**: 82 597 **Sol:** p.244

Problem 2.30 (USAMO 1997/2). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively are concurrent. **Hints**: 596 2 611

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. **Hints**: 49 159 134

Problem 2.32 (USAMO 1998/2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC. **Hints**: 659 355 482

Problem 2.33 (IMO 2000/1). Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **Hints**: 17 174

Problem 2.34 (Canada 1990/3). Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$. **Hints**: 1 414 440 **Sol:** p.245

Problem 2.35 (IMO 2009/2). Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$. **Hints**: 78 544 346

Problem 2.36. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that (AOD), (BOE), and (COF) intersect at point X other than O. **Hints**: 553 79 **Sol:** p.245

Problem 2.37 (Canada 2007/5). Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let ω, ω_1, ω_2, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

(a) Prove that ω_1, ω_2, and ω_3 intersect in a common point.
(b) Show that lines PD, QE, and RF are concurrent. **Hints**: 376 548 660

Problem 2.38 (Iran TST 2011/1). In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$. **Hints**: 297 495 154 **Sol:** p.246

CHAPTER 3

Lengths and Ratios

As one, who versed in geometric lore, would fain
Measure the circle — Dante, *The Divine Comedy*

3.1 The Extended Law of Sines

Aside from angles and similar triangles, one way to relate angles to lengths is through the **law of sines**. A more thorough introduction to the true power of trigonometry occurs in Section 5.3, but we see that it already proves useful here in our study of lengths.

Theorem 3.1 (The Extended Law of Sines). *In a triangle ABC with circumradius R, we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

This so-called "extended form" contains the final clause of $2R$ at the end. It has the advantage that it makes the symmetry more clear (if $\frac{a}{\sin A} = 2R$ is true, then the other parts follow rather immediately). The extended form also gives us a hint of a direct proof:

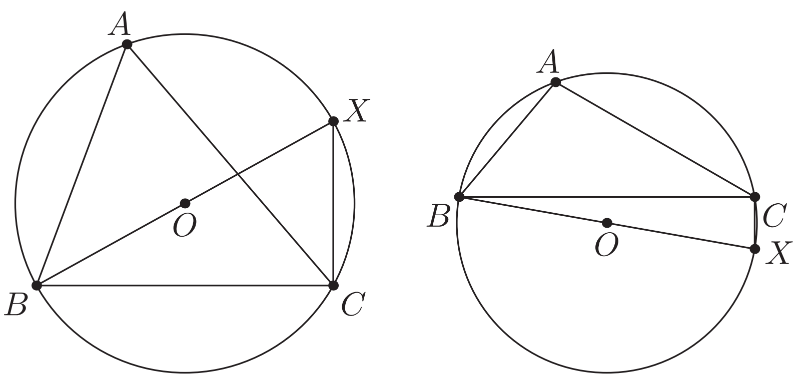

Figure 3.1A. Proving the law of sines.

Proof. As discussed above we only need to prove $\frac{a}{\sin A} = 2R$. Let $\overline{BX}$ be a diameter of the circumcircle, as in Figure 3.1A. Evidently $\measuredangle BXC = \measuredangle BAC$. Now consider triangle

BXC. It is a right triangle with $BC = a$, $BX = 2R$, and either $\angle BXC = A$ or $\angle BXC = 180^\circ - A$ (depending on whether $\angle A$ is acute). Either way,

$$\sin A = \sin \angle BXC = \frac{a}{2R}$$

and the proof ends here. □

The law of sines will be used later to provide a different form of the upcoming Ceva's theorem, namely Theorem 3.4.

Problem for this Section

Theorem 3.2 (Angle Bisector Theorem). *Let ABC be a triangle and D a point on $\overline{BC}$ so that $\overline{AD}$ is the internal angle bisector of $\angle BAC$. Show that*

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Hint: 417

3.2 Ceva's Theorem

In a triangle, a **cevian** is a line joining a vertex of the triangle to a point on the interior* of the opposite side. A natural question is when three cevians of a triangle are concurrent. This is answered by Ceva's theorem.

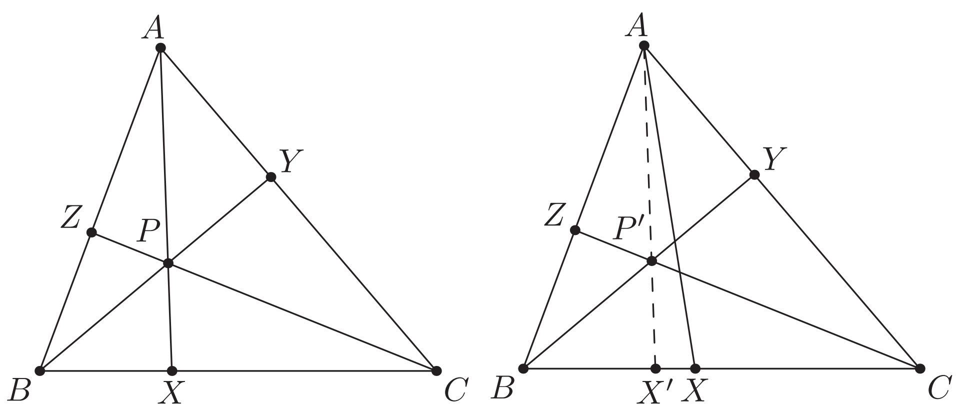

Figure 3.2A. Three cevians are concurrent as in Ceva's theorem.

Theorem 3.3 (Ceva's Theorem). *Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be cevians of a triangle ABC. They concur if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

The proof is by areas: we use the fact that if two triangles share an altitude, the ratio of the areas is the ratio of their bases. This trick is very useful in general.

* Some authors permit cevians to land on points on the *extensions* of the opposite side as well. For this chapter we assume cevians lie in the interior of the triangle unless otherwise specified.

Proof. Let us first assume the cevians concur at P, and try to show the ratios multiply to 1. Since $\triangle BAX$ and $\triangle XAC$ share an altitude, as do $\triangle BPX$ and $\triangle XPC$, we derive

$$\frac{BX}{XC} = \frac{[BAX]}{[XAC]} = \frac{[BPX]}{[XPC]}.$$

Now we are going to use a little algebraic trick: if $\frac{a}{b} = \frac{x}{y}$, then $\frac{a}{b} = \frac{x}{y} = \frac{a+x}{b+y}$. For example, since $\frac{4}{6} = \frac{10}{15}$, both are equal to $\frac{4+10}{6+15} = \frac{14}{21}$. Applying this to the area ratios yields

$$\frac{BX}{XC} = \frac{[BAX] - [BPX]}{[XAC] - [XPC]} = \frac{[BAP]}{[ACP]}.$$

But now the conclusion is imminent, since

$$\frac{CY}{YA} = \frac{[CBP]}{[BAP]} \text{ and } \frac{AZ}{ZB} = \frac{[ACP]}{[CBP]}$$

whence multiplying gives the desired $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

Now how do we handle the other direction? Dead simple with phantom points. Assume $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are cevians with

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Let $\overline{BY}$ and $\overline{CZ}$ intersect at P', and let ray AP' meet $\overline{BC}$ at X' (right half of Figure 3.2A). By our work already done, we know that

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Thus $\frac{BX'}{X'C} = \frac{BX}{XC}$, which is enough to imply $X = X'$. □

The proof above illustrated two useful ideas—the use of area ratios, and the use of phantom points.

As you might guess, Ceva's theorem is extremely useful for showing that three lines are concurrent. It can also be written in a trigonometric form.

Theorem 3.4 (Trigonometric Form of Ceva's Theorem). *Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be cevians of a triangle ABC. They concur if and only if*

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1.$$

The proof is a simple exercise—just use the law of sines.

With this, the existence of the orthocenter, the incenter, and the centroid are all totally straightforward. For the orthocenter[†], we compute

$$\frac{\sin(90^\circ - B)\sin(90^\circ - C)\sin(90^\circ - A)}{\sin(90^\circ - C)\sin(90^\circ - A)\sin(90^\circ - B)} = 1.$$

[†] Actually we need to handle the case where $\triangle ABC$ is obtuse separately, since in that case two of the altitudes fall outside the triangle. We develop the necessary generalization in the next section, when we discuss directed lengths in Menelaus's theorem.

For the incenter, we compute

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}=1.$$

We could also have used the angle bisector theorem in the standard form of Ceva's theorem, giving

$$\frac{c}{b}\frac{a}{c}\frac{b}{a}=1.$$

Finally, for the centroid we have

$$\frac{1}{1}\frac{1}{1}\frac{1}{1}=1$$

and we no longer have to take the existence of our centers for granted!

Problems for this Section

Problem 3.5. Show the trigonometric form of Ceva holds.

Problem 3.6. Let $\overline{AM}$, $\overline{BE}$, and $\overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

3.3 Directed Lengths and Menelaus's Theorem

The analogous form of Ceva's theorem is called Menelaus's theorem, which specifies when three points on the sides of a triangle (or their extensions) are collinear.

Theorem 3.7 (Menelaus's Theorem). *Let X, Y, Z be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then X, Y, Z are collinear if and only if*

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=-1$$

where the lengths are directed.

Here we have introduced ratios of **directed lengths**. Given collinear points A, Z, B, we say that the ratio $\frac{AZ}{ZB}$ is positive if Z lies between A and B, and negative otherwise. (This is much the same idea as the signs we used in defining the power of a point.) We always say explicitly when lengths are taken to be directed.

Notice the similarity to Ceva's theorem. The use of -1 instead of 1 is important—for if X, Y, Z each lie in the interiors of the sides, it is impossible for the three to be collinear!

Essentially the directed lengths are simply encoding two cases of Menelaus's theorem: when either one or three of $\{X, Y, Z\}$ lie outside the corresponding side. It is easy to check that the sign of the directed ratio is negative precisely in these cases.

There are many proofs of Menelaus's theorem that we leave to other sources. The proof we give shows one direction; if the ratios multiply to -1, then the points are collinear. (The other direction then follows using phantom points.) It is inspired by a proof to Monge's theorem (Theorem 3.22), and it is so surprising that we could not resist including it.

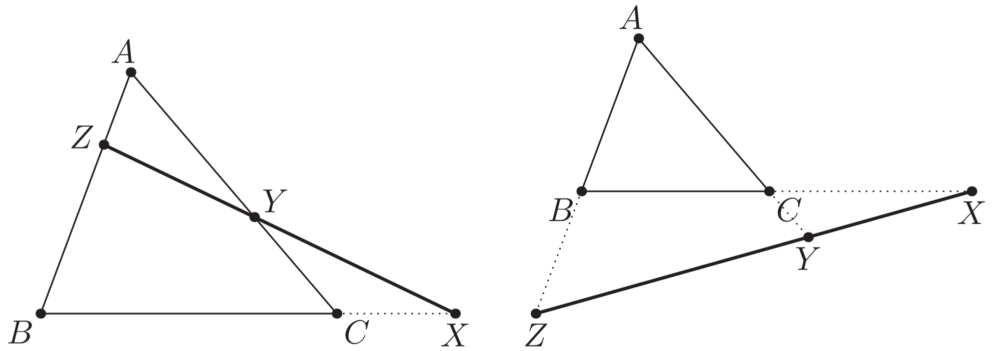

Figure 3.3A. The two cases of Menelaus's theorem.

Proof. First, suppose that the points X, Y, Z lie on the sides of the triangle in such a way that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

Then it is possible to find nonzero real numbers p, q, r for which

$$\frac{q}{r} = -\frac{BX}{XC}, \quad \frac{r}{p} = -\frac{CY}{YA}, \quad \frac{p}{q} = -\frac{AZ}{ZB}.$$

Now we go into three dimensions! Let $\mathcal{P}$ be the plane of triangle ABC (this page) and construct point A_1 such that $\overline{A_1A} \perp \mathcal{P}$ and $AA_1 = p$; we take A_1 to be above the page if $p > 0$ and below the page otherwise. Now define B_1 and C_1 analogously, so that $BB_1 = q$ and $CC_1 = r$.

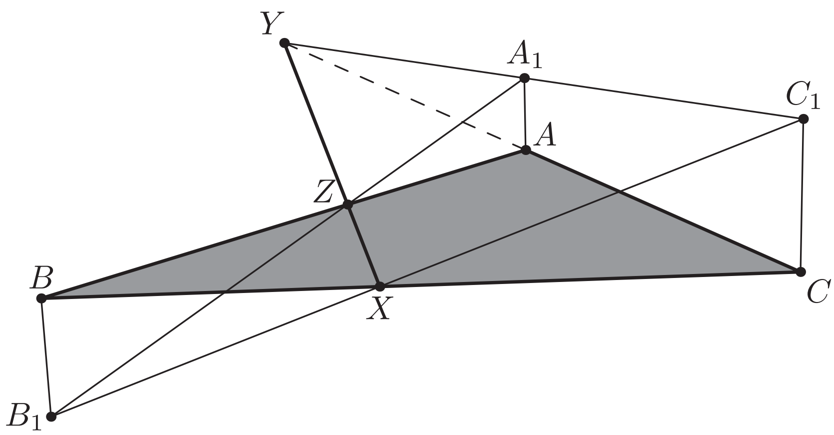

Figure 3.3B. The 3D proof of Menelaus's theorem.

One can easily check (say, by similar triangles) that the points B_1, C_1, and X are collinear. Indeed, just consider the right triangles C_1CX and B_1BX, and note the ratios of the legs. Similarly, line A_1B_1 passes through Z and A_1C_1 passes through Y.

But now consider the plane $\mathcal{Q}$ of the triangle $A_1B_1C_1$. The intersection of planes $\mathcal{P}$ and $\mathcal{Q}$ is a line. However, this line contains the points X, Y, Z, so we are done. $\square$

It also turns out that Ceva's theorem (as well as its trigonometric form) can be generalized using directed lengths. We can write this in the following manner. This should be taken as the full form of Ceva's theorem.

Theorem 3.8 (Ceva's Theorem with Directed Lengths). *Let ABC be a triangle and X, Y, Z be points on lines BC, CA, AB distinct from its vertices. Then lines AX, BY, CZ are concurrent if and only if*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1$$

where the ratios are directed.

The condition is equivalent to

$$\frac{\sin \angle BAX \sin \angle CBY \sin \angle ACZ}{\sin \angle XAC \sin \angle YBA \sin \angle ZCB} = 1$$

where either exactly one or exactly three of X, Y, Z lie strictly inside sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Because exactly two altitudes land outside the sides in an obtuse triangle, this generalization lets us complete the proof that the orthocenter exists for obtuse triangles. (What about for right triangles?)

3.4 The Centroid and the Medial Triangle

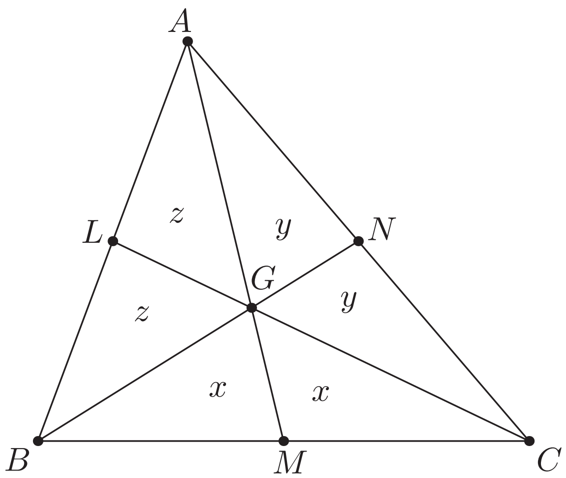

Figure 3.4A. Area ratios on the centroid of a triangle.

We can say even more about the centroid than just its existence by again considering area ratios. Consider Figure 3.4A, where we have added the midpoints of each of the sides (the triangle they determine is called the **medial triangle**). Notice that

$$1 = \frac{BM}{MC} = \frac{[GMB]}{[CMG]}$$

as discussed before in our proof of Ceva's theorem. Consequently $[GMB] = [CMG]$ and so we mark their areas with an x in Figure 3.4A. We can similarly define y and z.

But now, by the exact same reasoning,

$$1 = \frac{BM}{MC} = \frac{[AMB]}{[CMA]} = \frac{x + 2z}{x + 2y}.$$

Hence $y = z$. Analogous work gives $x = y$ and $x = z$. So that means the six areas of the triangles are all equal.

In that vein, we deduce

$$\frac{AG}{GM} = \frac{[GAB]}{[MGB]} = \frac{2z}{x} = 2.$$

This yields an important fact concerning the centroid of the triangle.

Lemma 3.9 (Centroid Division). *The centroid of a triangle divides the median into a* $2:1$ *ratio.*

Just how powerful can area ratios become? Answer: you can build a whole coordinate system around them. See Chapter 7.

3.5 Homothety and the Nine-Point Circle

First of all, what is a homothety? A **homothety** or **dilation** is a special type of similarity, in which a figure is dilated from a center. See Figure 3.5A.

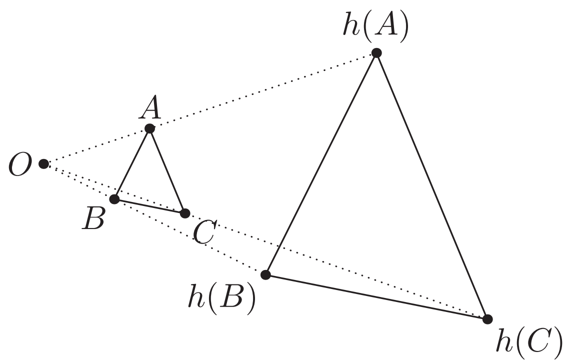

Figure 3.5A. A homothety h with center O acting on ABC.

More formally, a homothety h is a transformation defined by a center O and a real number k. It sends a point P to another point $h(P)$, multiplying the distance from O by k. The number k is the **scale factor**. It is important to note that k can be negative, in which case we have a **negative homothety**. See Figure 3.5B.

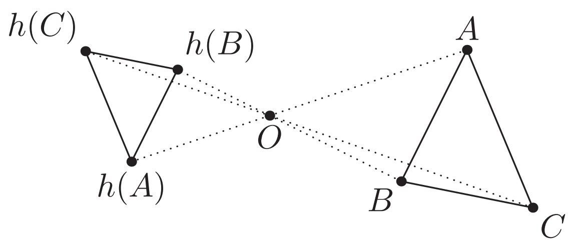

Figure 3.5B. A negative homothety with center O.

In other words, all this is a fancy special case of similar triangles.

Homothety preserves many things, including but not limited to tangency, angles (both vanilla and directed), circles, and so on. They do not preserve length, but they work well enough: the lengths are simply all multiplied by k.

Furthermore, given noncongruent parallel segments $\overline{AB}$ and $\overline{XY}$ (what happens if $AB = XY$?), we can consider the intersection point O of lines AX and BY. This is the center of a homothety sending one segment to the other. (As is the intersection of lines AY and BX—one of these is negative.) As a result, parallel lines are often indicators of homotheties.

A consequence of this is the following useful lemma.

Lemma 3.10 (Homothetic Triangles). *Let ABC and XYZ be non-congruent triangles such that $\overline{AB} \parallel \overline{XY}$, $\overline{BC} \parallel \overline{YZ}$, and $\overline{CA} \parallel \overline{ZX}$. Then lines AX, BY, CZ concur at some point O, and O is a center of a homothety mapping $\triangle ABC$ to $\triangle XYZ$.*

Convince yourself that this is true. The proof is to take a homothety h with $X = h(A)$ and $Y = h(B)$ and then check that we must have $Z = h(C)$.

One famous application of homothety is the so-called **nine-point circle**. Recall Lemma 1.17, which states that the reflection of the orthocenter over $\overline{BC}$, as well as the reflection over the midpoint of $\overline{BC}$, lies on (ABC). In Figure 3.5C, we have added in the reflections over the other sides as well.

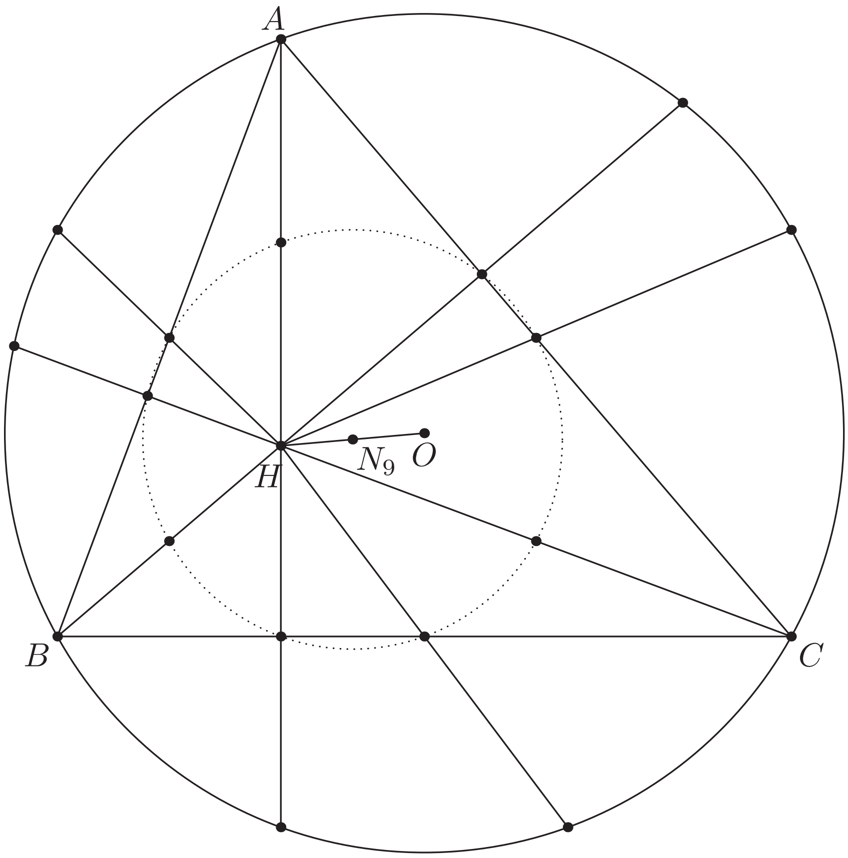

Figure 3.5C. The nine-point circle.

We now have nine points on (ABC) with center O, the three reflections of H over the sides, the three reflections of H over the midpoints, and the vertices of the triangle themselves.

Let us now take a homothety h at H (meaning with center H) and with scale factor $\frac{1}{2}$. This brings all the reflections back onto the sides of ABC, while also giving us as an added bonus the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$. In addition, O gets mapped to the midpoint of $\overline{OH}$, say N_9.

On the other hand homothety preserves circles, so astonishingly enough, these nine points remain concyclic. We even know the center of the circle—it is the image $h(O) = N_9$, called the **nine-point center**. We even know the radius! It is just half of the circumradius (ABC). This circles is called the nine-point circle.

Lemma 3.11 (Nine-Point Circle). *Let ABC be a triangle with circumcenter O and orthocenter H, and denote by N_9 the midpoint of $\overline{OH}$. Then the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, as well as the feet of the altitudes of $\triangle ABC$, lie on a circle centered at N_9. Moreover, the radius of this circle is half the radius of (ABC).*

We will see several more applications of homothety in Chapter 4, but this is one of the most memorable. A second application is the **Euler line**—the circumcenter, orthocenter, and centroid are collinear as well! We leave this famous result as Lemma 3.13; see Figure 3.5D.

Problems for this Section

Problem 3.12. Give an alternative proof of Lemma 3.9 by taking a negative homothety. **Hints**: 360 165 348

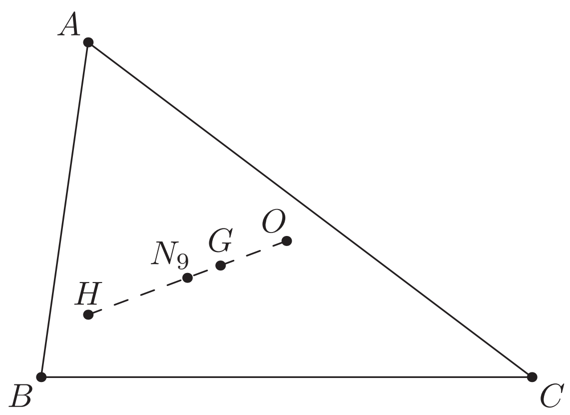

Figure 3.5D. The Euler line of a triangle.

Lemma 3.13 (Euler Line). *In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides $\overline{OH}$ in a $2:1$ ratio.* **Hints**: 426 47 314

3.6 Example Problems

Our first example is from the very first European Girl's Math Olympiad. It is a good example of how recognizing one of our configurations (in this case, the reflections of the orthocenters) can lead to an elegant solution.

Example 3.14 (EGMO 2012/7). Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of $\overline{BC}$ from A. Let L be the reflection of K across $\overline{AB}$, and let M be the reflection of K across $\overline{BC}$. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM, and BC are concurrent.

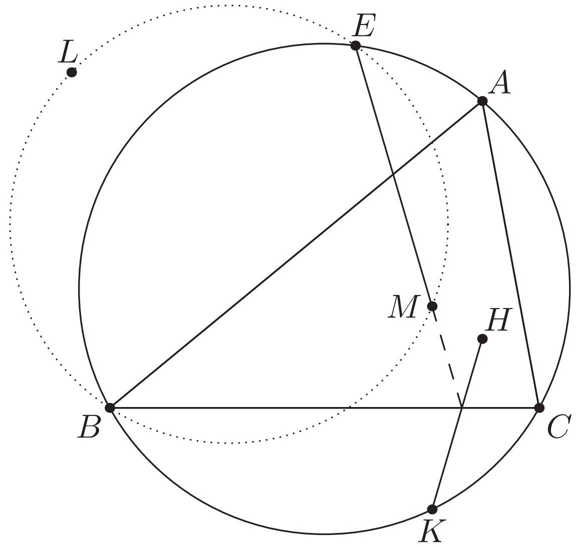

Figure 3.6A. From the first European Girl's Olympiad.

Upon first reading the problem, there are two observations we can make about it.

1. There are a lot of reflections.
2. The orthocenter does not do anything until the last sentence, when it magically appears as the endpoint of one of the concurrent lines.

This is a pretty tell-tale sign. What does the orthocenter have to do with reflections and the circumcircle? We need to tie in the orthocenter *somehow*, otherwise it is just floating in the middle of nowhere. How do we do this?

These questions motivate us to reflect H over $\overline{BC}$ and $\overline{AB}$ to points H_A and H_C, corresponding to the reflections of K across these segments. This move incorporates both the observations above. At this point we realize that $\overline{MH_A}$ and $\overline{HK}$ concur on $\overline{BC}$ for obvious reasons. So the problem is actually asking to show that H_A, M, and E are collinear. This is certainly progress.

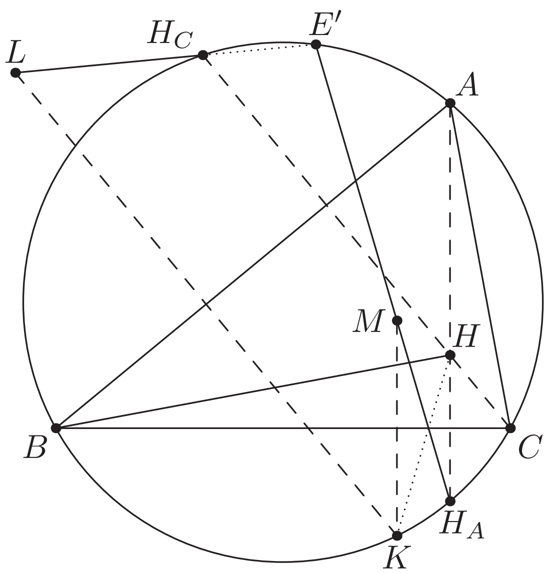

Figure 3.6B. Adding in some reflections.

At this point we can instead let E' be the intersection of $H_A M$ with Γ and try to show that $BLE'M$ is concyclic. We are motivated to use phantom points to handle collinearity (since "concyclic" is easier to show), and we choose E because H_A and M are simpler—they are just reflections of given points. (Of course, it is probably possible to rewrite the proof without phantom points.) In any case, it suffices to prove $\measuredangle LE'M = \measuredangle LBM$.

However, we can compute $\measuredangle LBM$ easily. It is just

$$\measuredangle LBK + \measuredangle KBM = 2(\measuredangle ABK + \measuredangle KBC) = 2\measuredangle ABC.$$

So now we have reduced this to showing that $\measuredangle LE'M = 2\measuredangle ABC$.

Examining the scaled diagram closely suggests that L, H_C, and E' might be collinear. Is this true? It would sure seem so. To see how useful our conjecture might be, we quickly conjure

$$\measuredangle H_C E' H_A = \measuredangle H_C B H_A = 2\measuredangle ABC.$$

Thus the desired conclusion is actually equivalent to showing these three points are collinear. Now we certainly want to establish this.

How do we go about proving this? Angle chasing seems the most straightforward. It would suffice to prove that $\measuredangle LH_C B = \measuredangle E'H_C B$; the latter is equal to $\measuredangle E'H_A B$, which by symmetry happens to equal $\measuredangle BHK$. So we need $\measuredangle LH_C B = \measuredangle BHK$—which is clear by symmetry.

Solution to Example 3.14. Let H_A and H_C be the reflections of H across $\overline{BC}$ and $\overline{BA}$, which lie on Γ. Let E' be the second intersection of line $H_A M$ with Γ. By construction, lines $E'M$ and HK concur on $\overline{BC}$. First, we claim that L, H_C, and E' are collinear. By reflections,

$$\measuredangle LH_C B = -\measuredangle KHB = \measuredangle MH_A B$$

and

$$\measuredangle MH_A B = \measuredangle E'H_A B = \measuredangle E'H_C B$$

as desired. Now,

$$\measuredangle LE'M = \measuredangle H_C E' H_A = \measuredangle H_C B H_A = 2\measuredangle ABC$$

and

$$\measuredangle LBM = \measuredangle LBK + \measuredangle KBM = 2\measuredangle ABK + 2\measuredangle KBC = 2\measuredangle ABC$$

so B, L, E', M are concyclic. Hence $E = E'$ and we are done. □

The second example is similar in spirit.

Example 3.15 (Shortlist 2000/G3). Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

The weird part of this problem is the sum condition. Why $OD + DH = OE + EH = OF + FH$? The good news is that at least we can (try to) pick the points D, E, F. So we focus on using this to get rid of the strange condition. Are there any choices of D, E, F that readily satisfy the condition, and which induce concurrent cevians?

Having a ruler and compass is important here. After you make a guess for the points D, E, F, you better make sure that the three lines look concurrent. It is helpful to have more than one diagram for this.

One guess might be to use orthocenter reflections again. If we let H_A, H_B, H_C denote the reflections, then $OD + DH_A = OE + EH_B = OF + FH_C$. Hence we can just pick let D be the intersection of $\overline{OH_A}$ and $\overline{BC}$, and define E and F similarly. Then $OD + DH_A = OE + EH_B = OF + FH_C = R$, where R is the circumradius of $\triangle ABC$.

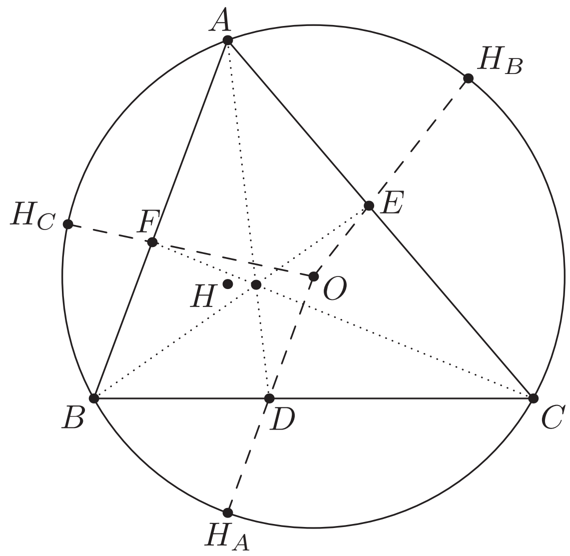

Figure 3.6C. Reflecting the orthocenter again for Example 3.15.

Now the moment of truth—are we lucky enough that the cevians concur? The computer-generated Figure 3.6C probably gives it away, but draw a diagram or two and convince yourself. This is how you check if you are going in the right direction on a contest.

Once convinced of that, we are in good shape. We just need to show that the cevians concur. Naturally, we fall back to Ceva's theorem for that. Unfortunately, we do not know much in the way of lengths (other than the carefully contrived $OD + DH = R$). Nor do we know much about the angles $\angle BAD$ and $\angle CAD$. So how else can we compute $\frac{BD}{CD}$? This is all we need, since once $\frac{BD}{CD}$ is found, we simply find the other two ratios in the same manner and multiply all three together. This product must be one, at which point we win.

The main idea now is to use the law of sines. Let us focus on triangles BDH_A and CDH_A. Because H_A was the reflection of an orthocenter, we know a lot about its angles. Specifically,

$$\measuredangle H_ABD = \measuredangle H_ABC = -\measuredangle HBC = 90^\circ - C$$

and

$$\measuredangle DH_AB = \measuredangle OH_AB = 90^\circ - \measuredangle BAH_A = 90^\circ - \measuredangle BAH = B$$

where $\measuredangle BH_AO = 90^\circ - \measuredangle BAH_A$ follows from Lemma 1.30. (Although I am mainly using directed angles here from force of habit; ABC is acute so this could likely be avoided.)

This is good, because the law of sines now lets us compute useful ratios. Noting that our angles were directed positively (that is, $\measuredangle H_ABD$ and $\measuredangle DH_AB$ both are counterclockwise), we can apply the law of sines to obtain

$$\frac{BD}{DH_A} = \frac{\sin \angle DH_AB}{\sin \angle H_ABD} = \frac{\sin B}{\cos C}.$$

The similar equation for $\triangle CDH_A$ is

$$\frac{CD}{DH_A} = \frac{\sin C}{\cos B}$$

and upon dividing we obtain

$$\frac{BD}{CD} = \frac{\sin B \cos B}{\sin C \cos C}.$$

Thus

$$\frac{CE}{EA} = \frac{\sin C \cos C}{\sin A \cos A}$$

and

$$\frac{AF}{FB} = \frac{\sin A \cos A}{\sin B \cos B}$$

and Ceva's theorem completes the solution.

A second alternative approach for obtaining the ratio $\frac{BD}{CD}$ involves the law of sines on triangle BOC. We present it below.

Solution to Example 3.15. Let H_A, H_B, H_C denote the reflections of H over $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Let D denote the intersection of $\overline{OH_A}$ and $\overline{BC}$. Evidently $OD + DH = OD + DH_A$ is the radius of (ABC). Hence if we select E and F analogously, we obtain $OD + DH = OE + EH = OF + FH$.

We now show that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent. Let R denote the circumradius of ABC. By the law of sines on $\triangle OBD$, we find that

$$\frac{BD}{R} = \frac{\sin \angle BOD}{\sin \angle BDO} = \frac{\sin 2\angle BAH_A}{\sin \angle BDO} = \frac{\sin 2B}{\sin \angle BDO}.$$

Similarly,

$$\frac{CD}{R} = \frac{\sin 2C}{\sin \angle CDO}$$

whence dividing gives

$$\frac{BD}{CD} = \frac{\sin 2B}{\sin 2C}.$$

It follows that

$$\frac{BD}{CD} \cdot \frac{CE}{EA} \cdot \frac{BF}{FA} = 1$$

and hence we are done by Ceva's theorem. □

What is the moral of the story here? First of all, good diagrams are really important for making sure what you are trying to prove is true. Secondly, flipping the orthocenter over the sides is a useful trick (though not the only one) for floating orthocenters that do not seem connected to anything else in the diagram. Thirdly, you should think of Ceva's theorem whenever you are going after a symmetric concurrency (as in this problem), since this lets you focus on just one third of the diagram and use symmetry on the other two-thirds. And finally, when you need ratios but only have angles, you can often make the connection via the law of sines.

3.7 Problems

Problem 3.16. Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur. The point of concurrency is the **Gergonne point**[‡] of triangle ABC. **Hint**: 683

Lemma 3.17. *In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.* **Hints**: 410 238 592 **Sol:** p.246

Problem 3.18. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Hints: 339 16 46

Problem 3.19 (Shortlist 2006/G3). Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$. **Hints**: 31 61 478 **Sol:** p.247

Problem 3.20 (BAMO 2013/3). Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC. **Hints**: 119 200 350

Problem 3.21 (USAMO 2003/4). Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$. **Hints**: 662 480 446

Theorem 3.22 (Monge's Theorem). *Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.* **Hints**: 102 48 **Sol:** p.247

[‡] Take note: the Gergonne point is not the incenter!

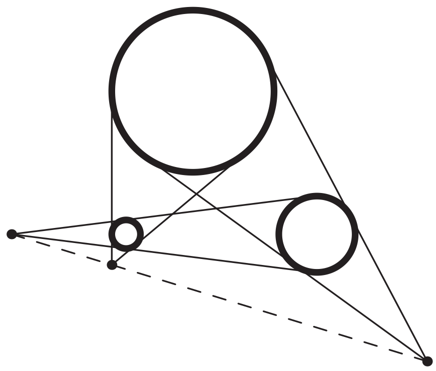

Figure 3.7A. Monge's theorem. The three points are collinear.

Theorem 3.23 (Cevian Nest). *Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be concurrent cevians of ABC. Let $\overline{XD}$, $\overline{YE}$, $\overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays AD, BE, CF concur.* **Hints**: 284 613 591 225 **Sol:** p.248

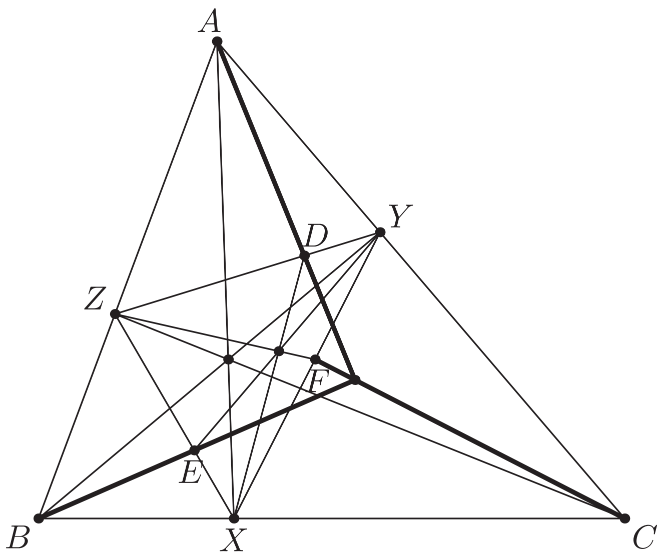

Figure 3.7B. Cevian nest

Problem 3.24. Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to $\overline{BC}$. Prove that K, G, X are collinear. **Hints**: 671 248 244

Problem 3.25 (USAMO 1993/2). Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic. **Hints**: 272 491 265

Problem 3.26 (EGMO 2013/1). The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled. **Hints**: 475 74 307 207 290 **Sol:** p.248

Problem 3.27 (APMO 2004/2). Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two. **Hints**: 599 152 598 545

Problem 3.28 (Shortlist 2001/G1). Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent. **Hints**: 618 665 383

Problem 3.29 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$. **Hints**: 459 570 148 **Sol:** p.249

Problem 3.30 (USAMO 2015/2). Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle. **Hints**: 533 501 116 639 418

CHAPTER 4

Assorted Configurations

There is light at the end of the tunnel, but it is moving away at speed c.

There are two ways to think about the configurations in this chapter. One is as a list of configurations to be memorized and recognized on contests. Another is as just a set of problems that frequently appear as subproblems (or superproblems) of olympiad problems. We prefer the second view, and have arranged this chapter accordingly.

4.1 Simson Lines Revisited

Let ABC be a triangle and P be any point, and denote by X, Y, Z the feet of the perpendiculars from P onto lines BC, CA, and AB. From Lemma 1.48 the points X, Y, Z are collinear if and only if P lies on (ABC). When P does lie on (ABC), this is called the **Simson line** of P with respect to ABC. We can say much more about this.

Denote by H the orthocenter of triangle ABC and let line PX meet (ABC) again at a point K, and let line AH intersect the Simson line at the point L. The completed figure is shown in Figure 4.1A.

We make a few synthetic observations.

Proposition 4.1. *Prove that the Simson line is parallel to $\overline{AK}$ in the notation of Figure 4.1A.* **Hints**: 390 151

Of course $\overline{XK} \parallel \overline{AL}$, and hence we discover $LAKX$ is a parallelogram.

Problem 4.2. Let K' be the reflection of K across $\overline{BC}$. Show that K' is the orthocenter of $\triangle PBC$. **Hint**: 521

We can now apply Lemma 3.17 to deduce that $AHPK'$ is a parallelogram. Using this, one can solve the next problem.

Problem 4.3. Show that $LHXP$ is a parallelogram. **Hint**: 97

From the above we can immediately deduce Lemma 4.4.

Lemma 4.4 (Simson Line Bisection). *Let ABC be a triangle with orthocenter H. If P is a point on (ABC) then its Simson line bisects $\overline{PH}$.*

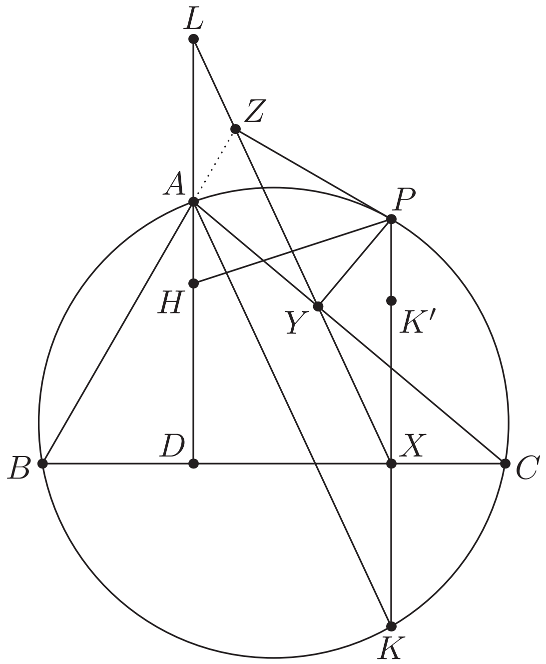

Figure 4.1A. Simson lines revisited.

Do not miss Simson lines when they appear. Contest problems that involve the Simson line usually only drop two of the altitudes and thus clandestinely construct the Simson line. Do not be fooled!

4.2 Incircles and Excircles

In Figure 4.2A we have drawn all three excenters of triangle ABC. Angle chasing gives an easy observation.

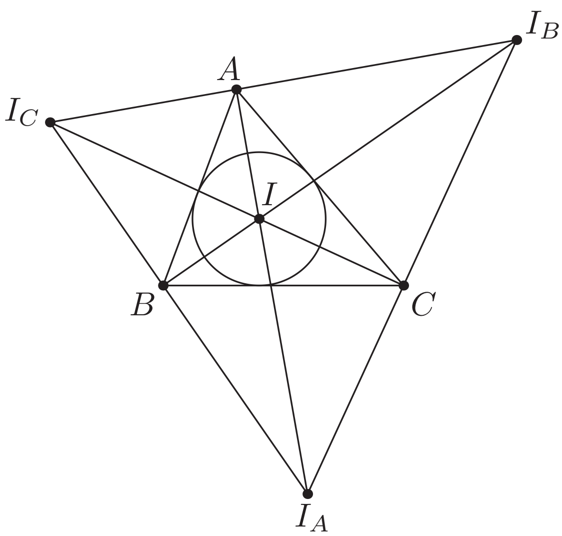

Figure 4.2A. The excenters of a triangle.

Problem 4.5. Check $\angle IAI_B = 90^\circ$ and $\angle IAI_C = 90^\circ$.

As a corollary, A lies on $\overline{I_B I_C}$. We also know (say, from Section 2.6) that the points A, I, and I_A are collinear. Actually $\overline{AI_A} \perp \overline{I_B I_C}$. Our observations can be summarized as follows.

Lemma 4.6 (Duality of Orthocenters and Excenters). *If I_A, I_B, I_C are the excenters of $\triangle ABC$, then triangle ABC is the orthic triangle of $\triangle I_A I_B I_C$, and the orthocenter is I.*

This duality is important to remember. The orthic triangle and excenters are "dual" concepts—they correspond exactly to each other. Problem writers sometimes phrase a problem stated more naturally in one framework with the other in an effort to make the problem artificially harder. Watch for this when it happens.

Problem 4.7. How are Lemma 1.18, Lemma 3.11, and Lemma 4.6 related? **Hint**: 458

Let us now concentrate further on a smaller part of the diagram. In Figure 4.2B we focus on just the A-excircle, tangent to $\overline{BC}$ at point X. We have drawn a line parallel to $\overline{BC}$ tangent again to the incircle at a point E. Suppose it intersects $\overline{AB}$ and $\overline{AC}$ at B' and C'. Evidently $\triangle AB'C'$ and $\triangle ABC$ are homothetic. But the incircle of $\triangle ABC$ is the A-excircle of $\triangle AB'C'$.

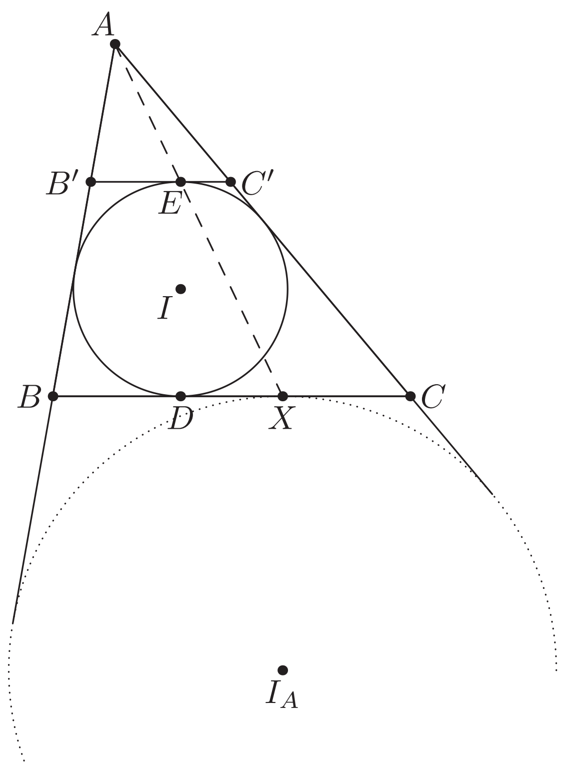

Figure 4.2B. The homothety between the incircle and A-excircle.

Problem 4.8. Prove that A, E, and X are collinear and that $\overline{DE}$ is a diameter of the incircle. **Hint**: 508

We also know that $BD = CX$, so we can actually phrase this statement without referring to the excircle.

Lemma 4.9 (The Diameter of the Incircle). *Let ABC be a triangle whose incircle is tangent to $\overline{BC}$ at D. If $\overline{DE}$ is a diameter of the incircle and ray AE meets $\overline{BC}$ at X, then $BD = CX$ and X is the tangency point of the A-excircle to $\overline{BC}$.*

Incircles and excircles often have dual properties. For example, check that the following is true as well.

Lemma 4.10 (Diameter of the Excircle). *In the notation of Lemma 4.9, suppose $\overline{XY}$ is a diameter of the A-excircle. Show that D lies on $\overline{AY}$.*

Problem for this Section

Problem 4.11. If M is the midpoint of $\overline{BC}$, prove that $\overline{AE} \parallel \overline{IM}$.

4.3 Midpoints of Altitudes

The results from the previous configuration extend to our next one. In Figure 4.3A we have removed the points B' and C' from Figure 4.2B and added an altitude $\overline{AK}$ with midpoint M. By Lemma 4.9 and Lemma 4.10, we already know that A, E, and X are collinear, as are A, D, and Y.

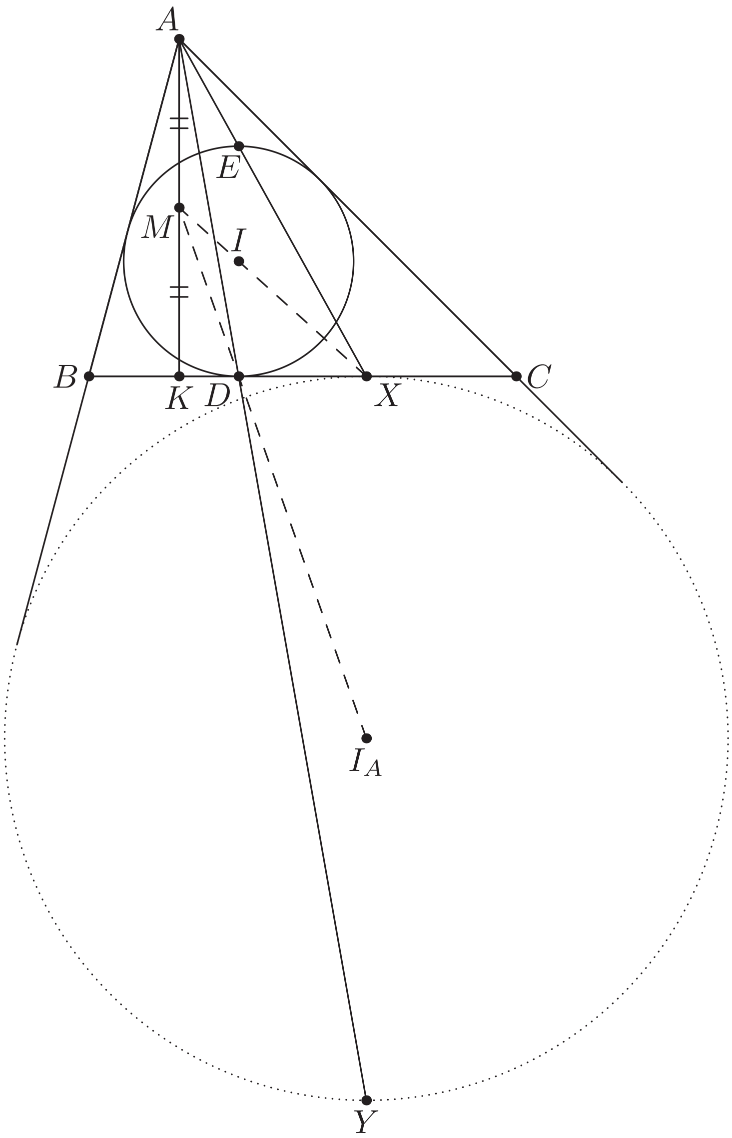

Figure 4.3A. Midpoints of altitudes.

Problem 4.12. Prove that points X, I, M are collinear. **Hints**: 138 175

Problem 4.13. Show that D, I_A, M are collinear. **Hint**: 336

We can restate these results as the following lemma.

Lemma 4.14 (Midpoint of Altitudes). *Let ABC be a triangle with incenter I and A-excenter I_A, and let D and X be the associated tangency points on $\overline{BC}$. Then lines DI_A and XI concur at the midpoint of the altitude from A.*

4.4 Even More Incircle and Incenter Configurations

Let DEF be the contact triangle of a triangle ABC, and consider the point X on $\overline{EF}$ such that $\overline{XD} \perp \overline{BC}$. The situation is shown in Figure 4.4A. The claim is that ray AX bisects $\overline{BC}$.

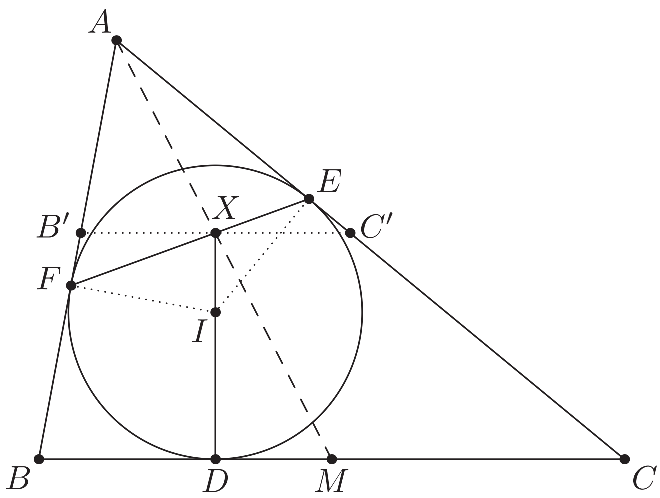

Figure 4.4A. The median intersects a side of the contact triangle.

Suppose we were trying to prove this. The key insight is that point M is kind of a distraction. We can eliminate it, along with $\overline{BC}$, by taking the line through X parallel to $\overline{BC}$ and considering a homothety. Let the line meet $\overline{AB}$ and $\overline{AC}$ again at B' and C'. Now it suffices to prove that X is the midpoint of $\overline{B'C'}$.

Problem 4.15. Show that I must lie on $(AB'C')$. **Hint**: 64

Problem 4.16. Prove that $XB' = XC'$. **Hint**: 470

Once we have these results, our next configuration is immediate.

Lemma 4.17 (An Incircle Concurrency). *Let ABC be a triangle with incenter I and contact triangle DEF. If M is the midpoint of $\overline{BC}$, then $\overline{EF}$, $\overline{AM}$ and ray DI concur.*

4.5 Isogonal and Isotomic Conjugates

This particular configuration is fairly straightforward.

Lemma 4.18 (Isogonal Conjugates). *Let ABC be a triangle and P any point not collinear with any of the sides. There exists a unique point P^* satisfying the relations*

$$\measuredangle BAP = \measuredangle P^*AC, \quad \measuredangle CBP = \measuredangle P^*BA, \quad \measuredangle ACP = \measuredangle P^*CB.$$

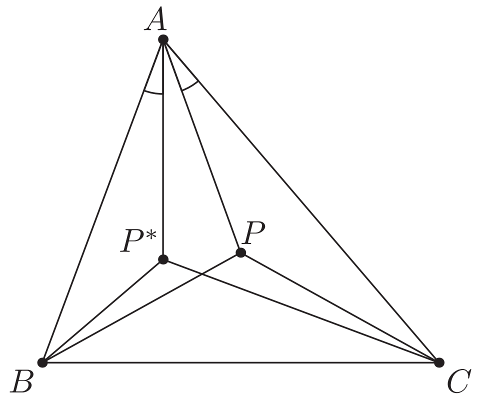

Figure 4.5A. P and P^* are isogonal conjugates.

The point P^* is called the **isogonal conjugate** of the point P. We also say line AP^* is **isogonal** to (or "is the isogonal of") line AP with respect to triangle ABC; however we often omit the phrase "with respect to triangle ABC" if the context is clear. In other words, two lines through A are isogonal if they are reflections over the angle bisector of $\angle A$.

A better way to phrase the lemma is the "buy two get one free" perspective, as in the exercise below.

Problem 4.19. Show that if two of the angle relations in Lemma 4.18 hold, then so does the third. **Hint**: 9

The **isotomic conjugate** is defined similarly. For a point P and triangle ABC, let X, Y, Z be the feet of the cevians through P. Let X' be the reflection of X about the midpoint of $\overline{BC}$ and define Y' and Z' similarly. Then the cevians $\overline{AX'}$, $\overline{BY'}$, and $\overline{CZ'}$ concur at a point P^t, the isotomic conjugate of P.

Problem 4.20. Prove that the cevians AX', BY', and CZ' concur as described above.

Problems for this Section

Problem 4.21. Check that if Q is the isogonal conjugate of P, then P is the isogonal conjugate of Q.

Theorem 4.22 (Isogonal Ratios). *Let D and E be points on $\overline{BC}$ so that $\overline{AD}$ and $\overline{AE}$ are isogonal. Then*

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$

Hint: 184

Problem 4.23. What is the isogonal conjugate of a triangle's circumcenter?

4.6 Symmedians

The isogonal of a median in a triangle is called a **symmedian**. The concurrency point of the three symmedians is the isogonal conjugate of the centroid, called the **symmedian point**.

Symmedians have tons of nice properties. We first show how they arise naturally.

Lemma 4.24 (Constructing the Symmedian). *Let X be the intersection of the tangents to (ABC) at B and C. Then line AX is a symmedian.*

The proof is a direct computation with the law of sines. Let M be the intersection of the isogonal of AX on $\overline{BC}$; we wish to prove that M is the midpoint of $\overline{BC}$.

Problem 4.25. Show that

$$\frac{BM}{MC} = \frac{\sin \angle B \sin \angle BAX}{\sin \angle C \sin \angle CAX} = 1.$$

Now let us describe several additional properties of symmedians.

Lemma 4.26 (Properties of the Symmedian). *Let ABC be a triangle, and let the tangents to its circumcircle at B and C meet at point X. Let $\overline{AX}$ meet (ABC) again at K and $\overline{BC}$ at D. Then $\overline{AD}$ is the A-symmedian and*

(a) $\overline{KA}$ is a K-symmedian of $\triangle KBC$.
(b) $\triangle ABK$ and $\triangle AMC$ are directly similar.
(c) We have

$$\frac{BD}{DC} = \left(\frac{AB}{AC}\right)^2.$$

(d) We have

$$\frac{AB}{BK} = \frac{AC}{CK}.$$

(e) (BCX) passes through the midpoint of $\overline{AK}$.
(f) $\overline{BC}$ is the B-symmedian of $\triangle BAK$, and the C-symmedian of $\triangle CAK$.
(g) $\overline{BC}$ is the interior angle bisector of $\angle AMK$, and $\overline{MX}$ is the exterior angle bisector.

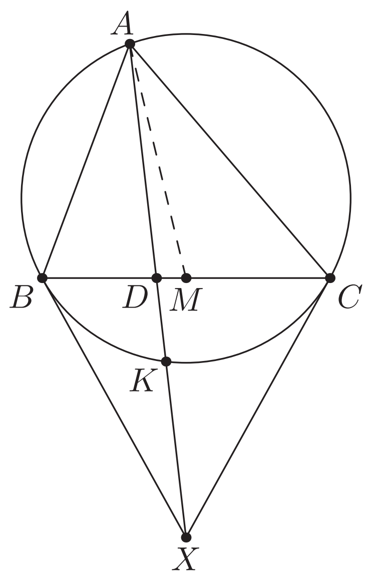

Figure 4.6A. The A-symmedian of a triangle

Here property (a) is obvious from the tangent construction, while (c) is a special case of Theorem 4.22. Properties (b) and (e) follow from straightforward angle chasing. The rest of the properties are described in the exercises. Extracting some of these properties yields the following.

Lemma 4.27 (Symmedians in Cyclic Quadrilaterals). *Let $ABCD$ be a cyclic quadrilateral. The following are equivalent.*

(a) $AB \cdot CD = BC \cdot DA$.
(b) $\overline{AC}$ *is an* A*-symmedian of* $\triangle DAB$.
(c) $\overline{AC}$ *is a* C*-symmedian of* $\triangle BCD$.
(d) $\overline{BD}$ *is a* B*-symmedian of* $\triangle ABC$.
(e) $\overline{BD}$ *is a* D*-symmedian of* $\triangle CDA$.

In Chapter 9, we learn that such a quadrilateral is called a **harmonic quadrilateral**, and possesses even more interesting properties.

Problems for this Section

Problem 4.28. Verify (d) of Lemma 4.26. **Hint**: 194

Problem 4.29. Show that (f) of Lemma 4.26 follows (with some effort) from (d). **Hints**: 190 628 584

Problem 4.30. Prove (g) of Lemma 4.26. **Hints**: 65 474

4.7 Circles Inscribed in Segments

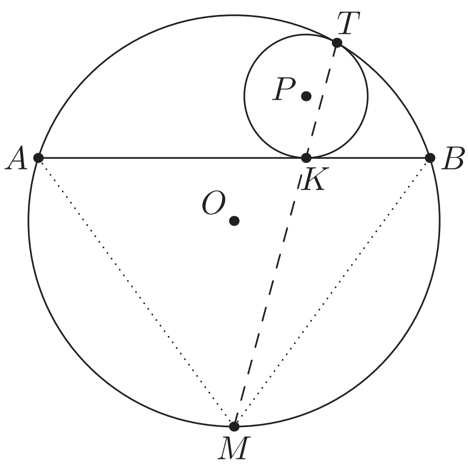

Figure 4.7A. A circle is inscribed in a segment.

Our next configuration involves a tangent circle. Let Ω be a circle with center O and a chord $\overline{AB}$, and consider a circle ω tangent internally to Ω at T and to $\overline{AB}$ at K. Let M denote the midpoint of the arc $\widehat{AB}$ not containing T. For no good reason, the region bounded by $\overline{AB}$ and the other arc $\widehat{AB}$ containing T is called a **segment**, hence the title of this section.

As the centers of ω and Ω are collinear with T (by tangency), it follows there is a homothety at T mapping ω to Ω.

Problem 4.31. Show that this homothety takes K to M, and in particular that T, K, and M are collinear.

Problem 4.32. Show that $\triangle TMB \sim \triangle BMK$.

The last implication gives that $MK \cdot MT = MB^2$. So, we deduce the following.

Lemma 4.33 (Circles Inscribed in Segments). *Let $\overline{AB}$ be a chord of a circle Ω. Let ω be a circle tangent to chord $\overline{AB}$ at K and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc $\widehat{AB}$ not containing T.*

Moreover, $MA^2 = MB^2$ is the power of M with respect to ω.

This configuration is even more straightforward with inversion, discussed in Chapter 8. A reader comfortable with inversion is encouraged to reconstruct the proof using a suitable inversion at M.

The above configuration extends naturally to the next one, shown in Figure 4.7B. Let C be another point on arc $\widehat{AB}$ containing T, and let D be a point on $\overline{AB}$ such that $\overline{CD}$ is tangent to ω at L.

The circle ω is called a **curvilinear incircle** of ABC. (As D varies along $\overline{AB}$, we obtain many curvilinear incircles, hence we refer to "a" curvilinear incircle. The next section discusses the special case $A = D$.) We claim that if I is the intersection of $\overline{CM}$ and $\overline{KL}$, then I is the incenter of $\triangle ABC$.

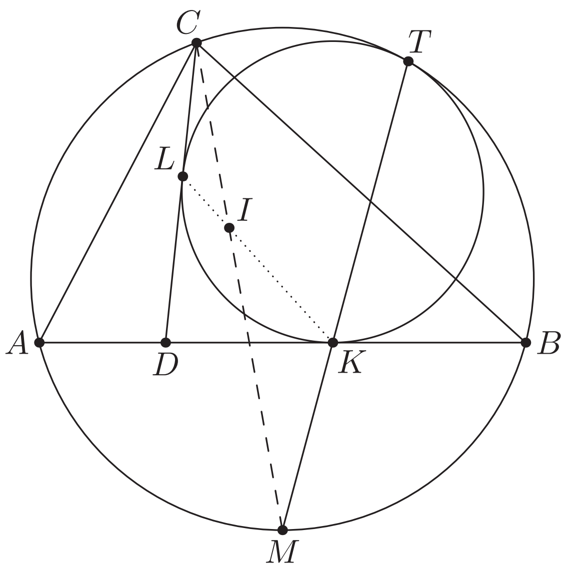

Figure 4.7B. More unusual tangent circles.

Problem 4.34. Prove that the points C, L, I, T are concyclic. **Hints**: 69 273 140

Problem 4.35. Show that $\triangle MKI \sim \triangle MIT$, and that the triangles are oppositely oriented. **Hints**: 472 236

Finally, how do we derive that I is the incenter? The similarity above gives that $MI^2 = MK \cdot MT$, but yet

$$MK \cdot MT = MA^2 = MB^2$$

by Lemma 4.33. Hence $MI = MA = MB$, and Lemma 1.18 establishes the configuration below.

Lemma 4.36 (Curvilinear Incircle Chords). *Let ABC be a triangle and D be a point on $\overline{AB}$. Suppose a circle ω is tangent to $\overline{CD}$ at L, $\overline{AB}$ at K, and also to (ABC). Then the incenter of ABC lies on line LK.*

4.8 Mixtilinear Incircles

The ***A*-mixtilinear incircle** of a triangle ABC is the circle internally tangent to (ABC), as well as to sides $\overline{AB}$ and $\overline{AC}$.

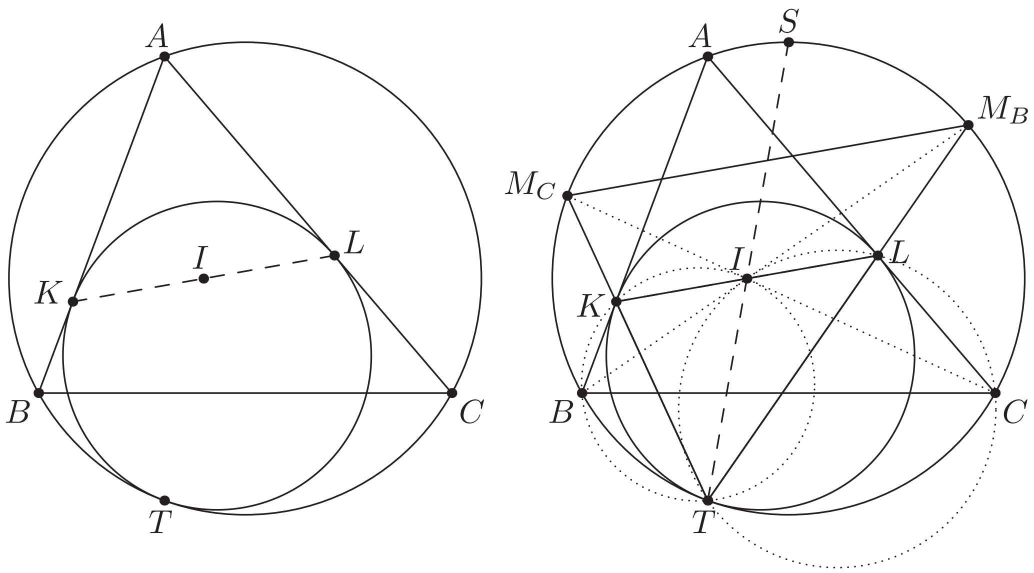

Figure 4.8A. An A-mixtilinear incircle.

Throughout this section, we let ω_A refer to this A-mixtilinear circle. Let T denote the tangency point of the ω_A with (ABC), and K and L the tangency points on $\overline{AB}$ and $\overline{AC}$. Taking $D = A$ in Lemma 4.36, we know that the incenter I of $\triangle ABC$ lies on $\overline{KL}$.

Problem 4.37. Using the fact that I lies on $\overline{KL}$, check that I is in fact the midpoint of $\overline{KL}$.

In Chapter 9 we give a nice alternative proof that I is the midpoint of $\overline{KL}$ using Pascal's theorem.

Let us see if we can learn anything interesting about the point T now. Let M_C and M_B be the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$. We of course already know (from Lemma 4.33) that T is the intersection of lines KM_C and LM_B. Now, extend line TI to meet the circumcircle of $\triangle ABC$ again at point S. The completed figure is show in Figure 4.8A.

Problem 4.38. Prove that $\angle ATK = \angle LTI$. **Hint**: 469

Problem 4.39. Prove that S is the midpoint of the arc $\widehat{BC}$ containing A. **Hint**: 342

Hence, we deduce that line TI passes through the midpoint of arc $\widehat{BC}$ not containing T. A second way to prove this is through angle chasing: one can show* that quadrilaterals $BKIT$ and $CLIT$ are cyclic since

$$\measuredangle IKT = \measuredangle LKT = \measuredangle M_B M_C T = \measuredangle M_B BT = \measuredangle IBT.$$

In any case this gives us $\measuredangle M_C TS = \measuredangle KTI = \measuredangle KBI = \measuredangle ABI$ for free, allowing us to establish the same conclusion as before.

In Chapter 8, we also prove (as part of Problem 8.31) that if E is the contact point of the A-excircle with $\overline{BC}$, then $\overline{AT}$ and $\overline{AE}$ are isogonal. Moreover, as Problem 4.49 we ask the reader to prove that the isogonal of $\overline{TA}$ with respect to $\triangle TBC$ passes through the contact point of the incircle at $\overline{BC}$. These additional results are exhibited in Figure 4.8B.

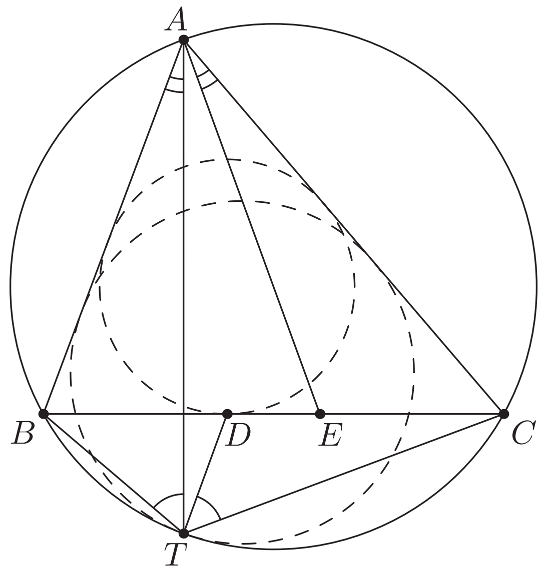

Figure 4.8B. Segments $\overline{AT}$ and $\overline{AE}$ are isogonal in $\triangle ABC$, while segments $\overline{TD}$ and $\overline{TA}$ are isogonal in $\triangle TBC$.

Combining the results in Figure 4.8A and Figure 4.8B into one big lemma:

Lemma 4.40 (Mixtilinear Incircles). *Let ABC be a triangle and let its A-mixtilinear circle be tangent to $\overline{AB}$, $\overline{AC}$, and (ABC) at K, L, and T, respectively. Denote by D and E the contact points of the incircle and A-excircle on $\overline{BC}$, respectively.*

(a) The midpoint I of $\overline{KL}$ is the incenter of $\triangle ABC$.
(b) Lines TK and TL pass through the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$ not containing T.
(c) Line TI passes through the midpoint of arc $\widehat{BC}$ containing A.
(d) The angles $\angle BAT$ and $\angle CAE$ are equal.
(e) The angles $\angle BTA$ and $\angle CTD$ are equal.
(f) Quadrilaterals $BKIT$ and $CLIT$ are concyclic.

For even more, see Lemma 7.42.

* Actually, we already proved this during our proof of Lemma 4.36.

4.9 Problems

These are not in any order—I cannot spoil the fun here!

Problem 4.41 (Hong Kong 1998). Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90^\circ$ and let H and K be the feet of the altitudes from Q to lines PR and PS. Prove that $\overline{HK}$ bisects $\overline{QS}$. **Hints**: 267 420

Problem 4.42 (USAMO 1988/4). Suppose $\triangle ABC$ is a triangle with incenter I. Show that the circumcenters of $\triangle IAB$, $\triangle IBC$, and $\triangle ICA$ lie on a circle whose center is the circumcenter of $\triangle ABC$. **Hint**: 249 **Sol:** p.249

Problem 4.43 (USAMO 1995/3). Given a nonisosceles, nonright triangle ABC, let O denote its circumcenter, and let A_1, B_1, and C_1 be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Point A_2 is located on the ray OA_1 so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent. **Hints**: 691 550 128

Problem 4.44 (USA TST 2014). Let ABC be an acute triangle and let X be a variable interior point on the minor arc $\widehat{BC}$. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line through P parallel to $\overline{XR}$. Prove that as X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point. **Hints**: 45 424 **Sol:** p.249

Problem 4.45 (USA TST 2011/1). In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$. Altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meet at orthocenter H. Points P and Q lie on segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR.
Hints: 124 317 26 **Sol:** p.250

Problem 4.46 (ELMO Shortlist 2012). Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of $\overline{AB}$. Another circle, ω_1 is tangent to Ω, ω, and $\overline{AB}$ at D, Z, and F respectively. Rays CD and AB meet at P. If $M \neq C$ is the midpoint of major arc AB, show that

$$\tan \angle ZEP = \frac{PE}{CM}.$$

Hints: 370 40 672 211

Problem 4.47 (USAMO 2011/5). Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$. **Hints**: 4 528

Problem 4.48 (Japanese Olympiad 2009). Triangle ABC is inscribed in circle Γ. A circle with center O is drawn, tangent to side BC at a point P, and internally tangent to the arc BC

of Γ not containing A at a point Q. Show that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$. **Hints**: 220 676 19

Problem 4.49. Let ABC be a triangle and let its incircle touch $\overline{BC}$ at D. Let T be the tangency point of the A-mixtilinear incircle with (ABC). Prove that $\angle BTA = \angle CTD$. **Hints**: 646 529 192 425

Problem 4.50 (Vietnam TST 2003/2). Let ABC be a scalene triangle with circumcenter O and incenter I. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes $\overline{AH}, \overline{BK}, \overline{CL}$, respectively. The incircle of triangle ABC touches the sides $\overline{BC}, \overline{CA}, \overline{AB}$ at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F, and OI are concurrent. **Hints**: 442 11 514 **Sol:** p.250

Problem 4.51 (Sharygin 2013). The incircle of $\triangle ABC$ touches $\overline{BC}, \overline{CA}, \overline{AB}$ at points A', B' and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$. **Hints**: 274 551 258

Problem 4.52 (APMO 2012/4). Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of $\overline{BC}$, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the ray MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold. **Hints**: 593 454 28 228 **Sol:** p.251

Problem 4.53 (Shortlist 2002/G7). The incircle Ω of the acute triangle ABC is tangent to $\overline{BC}$ at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of the segment $\overline{AD}$. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N **Hints**: 205 634 450 177 276

For a real challenge, check out Problem 11.19.