Mechanical Waves – I

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1 Introduction

What is a "wave"? It's difficult to give a precise answer as the concept is intrinsically somewhat vague; but here is a start:

▷ **Definition 1.** A wave is a disturbance of a continuous medium[1] that propagates with a fixed shape at a constant speed.

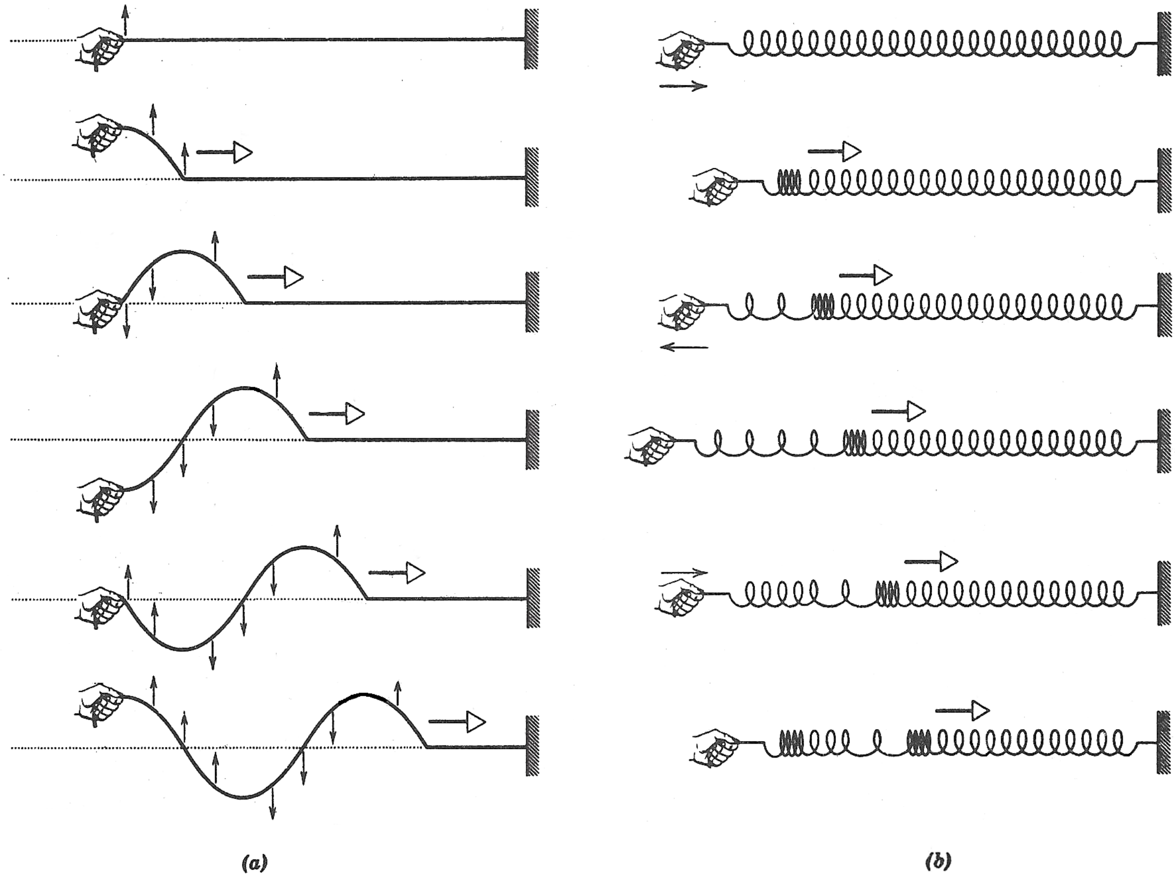


Figure 1: *(a)* In a transverse wave the particles of the medium (string) move in a direction that is perpendicular to the direction in which the wave itself propagates. *(b)* In a longitudinal wave the particles of the medium (spring) move in a direction that is parallel to the wave propagation.

To support wave motion, the medium needs to have some elastic properties. Because of this elastic property, once the disturbance is generated at some point of the medium it is transmitted from one point to the next point and consequently progresses though the medium. Note that the medium itself does not move as a whole along with the wave motion; the various parts of the medium move only over a limited path. For example, in water waves small floating objects like corks show that the actual motion of various parts of the water is slightly up and down and back and forth. Yet the water wave moves steadily along the water. And when they reach floating objects they set them in motion, thus transferring energy to them. Energy can thus be transmitted over considerable distances by wave motion.
If the particles of the medium conveying the wave motion move perpendicular to the direction of the wave itself, we call it a *transverse* wave (Figure 1(a)). If, however, the motion of the particles is back and forth along the direction of propagation, we have a *longitudinal* wave (Figure 1(b)),

[1]We are talking strictly about the *mechanical* waves. There are waves e.g. electromagnetic ones which do not require any medium to propagate

e.g. sound waves in a gaseous medium. But these are not the only kind of waves present. There are some waves like the water waves which are neither purely transverse nor purely longitudinal.

2 Waves on a String

2.1 Mathematical Requisite

The question now arises as to exactly how a wave should be represented mathematically. To keep things simple (and manageable) we restrict ourselves to the case of a *one dimensional* transverse wave. More specifically, we consider a long string stretched along the x-axis. We shall denote by y the displacement of a particle from its equilibrium position. Then y will be a function of both the position x of the particle and the moment of time t. At $t = 0$, we generate a disturbance by displacing the particles of the string in a transverse direction. The resulting initial shape of the string can thus be expressed by specifying y as some function of x, i.e. as $y = f(x, 0)$. Let this shape be as shown in the first of Figure 2. If the wave is to travel towards the positive x-axis with speed v without changing its shape, then we require that at a later time t it must be a replica of the initial shape but shifted by an amount vt, i.e.

$$f(x, t) = f(x - vt, 0) \tag{2.1}$$

which is shown in Figure 2. If instead the wave was travelling to the left, we should have $f(x, t) = f(x + vt, 0)$. So if in a function $f(x, t)$, the variables x and t appear in a specific

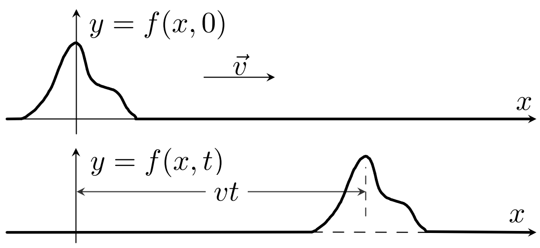


Figure 2: The shape at a later time is simply a shifted copy of the shape at the initial moment.

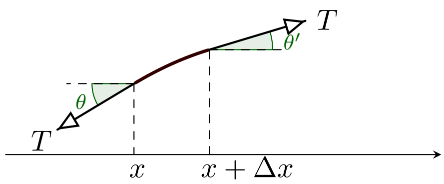


Figure 3: The free body diagram of the string in between x and $x + \Delta x$.

combination $x - vt$ or $x + vt$, then f can represent a wave travelling along the x-axis in a mathematical sense[2]. For example, if A and b are constants (with the appropriate units), then

$$f_1(x, t) = Ae^{-b(x-vt)^2}, \quad f_2(x, t) = A\sin(b(x - vt)), \quad f_3(x, t) = \frac{A}{1 + b(x - vt)^2}$$

all represent waves (with different shapes, of course), but

$$f_4(x, t) = Ae^{-b(bx^2+vt)}, \quad \text{and } f_5(x, t) = A\sin(bx)\cos(bvt)$$

do *not*.

2.2 The Wave Equation

The question now arises as to exactly *why* does a stretched string supports wave motion. It turns out that this is a consequence of the Newton's laws of motion. Suppose a very long string having a mass per unit length μ is stretched along the x axis under tension T. If it displaced

[2]For representing a physical wave, the function f must additionally be *bounded*, *continuous* and *differentiable* (smooth) as well.

from equilibrium (see Figure 3), the net force along the y direction on the portion lying between x and $x+\Delta x$ (here Δx is a small length) is:

$$F_y = T\sin\theta' - T\sin\theta$$

If the distortions of the string is not too large (which will usually be the case for the waves that we shall consider) the angles θ and θ' are small and so we can replace the sine with tangent:

$$F_y \approx T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \approx T\frac{\partial^2 y}{\partial x^2}\,\Delta x$$

In these equations, partial derivatives appear because y is a function of both x and t. Also, $\tan\theta$ is simply the slope of the tangent and so it has been replaced by $\partial y/\partial x$. Finally, the last equality follows from the fact that for small increment in the argument of a function, the increment in the function value can be approximated by the increment in the corresponding tangent.

Next, using Newton's second law, we could write $F_y = (\mu\Delta x)a_y = \mu\Delta x\frac{\partial^2 y}{\partial t^2}$ and so we have

$$T\frac{\partial^2 y}{\partial x^2}\,\Delta x = (\mu\Delta x)\frac{\partial^2 y}{\partial t^2},$$

which means that small disturbances on a stretched string satisfy the (partial) differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Setting

$$v = \sqrt{\frac{T}{\mu}} \tag{2.2}$$

(which, as we'll soon see, represents the speed of wave propagation) the above equation takes the form

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 y}{\partial t^2}} \tag{2.3}$$

This equation is known as the (classical) **wave equation**, because it admits as solutions all functions of the form

$$y = f(x,t) = g(x - vt) \tag{2.4}$$

(that is, all functions that depend on the variable x and t in the special combination $u \equiv x - vt$), and we have just learned that such functions represent waves propagating in the positive x direction with speed v. For Equation 2.4 means

$$\frac{\partial y}{\partial x} = \frac{\mathrm{d}g}{\mathrm{d}u}\frac{\partial u}{\partial x} = \frac{\mathrm{d}g}{\mathrm{d}u}, \quad \frac{\partial y}{\partial t} = \frac{\mathrm{d}g}{\mathrm{d}u}\frac{\partial u}{\partial t} = -v\frac{\mathrm{d}g}{\mathrm{d}u},$$

and so taking the second derivatives

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}g}{\mathrm{d}u}\right) = \frac{\mathrm{d}^2 g}{\mathrm{d}u^2}\frac{\partial u}{\partial x} = \frac{\mathrm{d}^2 g}{\mathrm{d}u^2},$$
$$\frac{\partial^2 y}{\partial t^2} = -v\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}g}{\mathrm{d}u}\right) = -v\frac{\mathrm{d}^2 g}{\mathrm{d}u^2}\frac{\partial u}{\partial t} = v^2\frac{\mathrm{d}^2 g}{\mathrm{d}u^2},$$

and so

$$\frac{\mathrm{d}^2 g}{\mathrm{d}u^2} = \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 y}{\partial t^2}$$

This last equality is precisely the Equation 2.3. Note that g can be *any* (differentiable) *function what so ever*. If the disturbance propagates without changing its shape, then it satisfies the wave equation.
But functions of the form $y = g(x - vt)$ are not the *only* solutions. The wave equation involves the *square* of v, so we can generate another set of solutions by simply changing the sign of the velocity: $y = h(x + vt)$. This of course represents a wave propagating along the *negative* x direction, and it is certainly reasonable (on physical grounds) that such solutions would be allowed. What is perhaps more surprising is that the *most general* solution to the wave equation is the sum of a wave along positive x direction and a wave along the negative x direction:

$$y = f(x,t) = g(x - vt) + h(x + vt) \tag{2.5}$$

This happens to be so because the wave equation is *linear* and so the **principle of superposition** is applicable: the sum of any two solutions is itself a solution. *Every* solution to the wave equation can be expressed in this form.
Like the simple harmonic motion equation, the wave equation is ubiquitous in physics. If something is vibrating, the simple harmonic equation is almost certainly responsible (at least, for small amplitudes), and if something is waving (whether in context of mechanics or acoustics, optics or oceanography), the wave equation (perhaps with some decorations) is bound to be involved.

2.3 Sinusoidal Waves

Of all possible wave forms, the sinusoidal one

$$y = f(x,t) = A\cos\big(k(x - vt) + \delta\big) \tag{2.6}$$

is (for good reasons) the most familiar. Figure 4 shows this wave at $t = 0$. A is the *amplitude* of

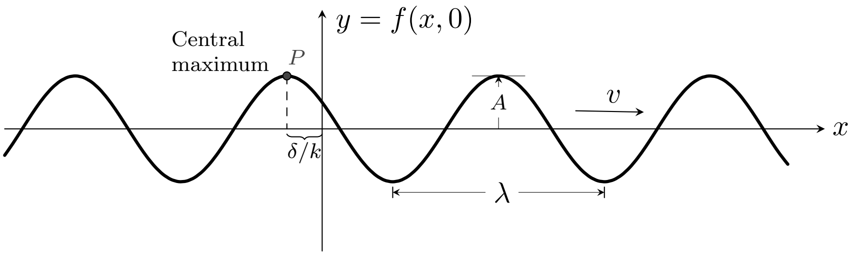


Figure 4: A sinusoidal wave at $t = 0$.

the wave. It is always positive and represents the maximum displacement from equilibrium. The argument of the cosine is called the *phase*, and δ is the *phase constant*. Obviously, one can add any integer multiple of 2π to δ without changing y; ordinarily the value in the range $0 \le \delta \le 2\pi$ is used. Notice that at $x = vt - \delta/k$, the phase is zero; let's call this the "central maximum". If $\delta = 0$, the central maximum passes the origin at $t = 0$; more generally, δ/k is the distance by which the central maximum and hence the entire wave is "delayed". Finally, k is the *wave number*: it is related to the *wavelength* λ by the equation

$$k = \frac{2\pi}{\lambda}, \tag{2.7}$$

for when x advances by $2\pi/k$, the cosine executes one complete cycle.
As time passes, the entire wave proceeds to the right, at speed v. At any fixed point x, the string vibrates up and down simple harmonically, undergoing one full cycle in a *period*

$$T = \frac{2\pi}{kv} \tag{2.8}$$

The *frequency* ν which is the number of oscillations per unit time is

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda} \tag{2.9}$$

For our purpose, a more convenient unit is the *angular frequency* ω, so called because in the analogous case of uniform circular motion it represents the number of radians swept out per unit time:

$$\omega = 2\pi\nu = kv \tag{2.10}$$

The sinusoidal waves represented by Equation 2.6 is customarily written in terms of ω, rather than v, as

$$y = A\cos(kx - \omega t + \delta) \tag{2.11}$$

A sinusoidal wave of wave number k and angular frequency ω travelling along the *negative* x-axis would be written

$$y = A\cos(kx + \omega t - \delta) \tag{2.12}$$

The sign of the phase constant is chosen for consistency with our previous convention that δ/k shall represent the distance by which the wave is "delayed". Since the wave is now moving to the *left*, a delay means a shift to the *right*. Thus at $t = 0$, the central maximum is located at a distance δ/k to the right of the origin. Because the cosine is an even function, we can just as well write the Equation 2.12 as:

$$y = A\cos(-kx - \omega t + \delta) \tag{2.13}$$

Comparison with Equation 2.11 reveals that, in effect, we could simply *change the sign of* k to produce a wave with the same amplitude, frequency, wavelength, and phase constant but travelling in opposite direction.

2.4 Power and Intensity in Waves

As has already been said, waves can transmit energy over long distances. Let us obtain the *power* transmitted by a wave moving along a long string stretched with tension T along the positive x-axis. Let v be the wave velocity.
Consider the point at x on the string as shown in Figure 5. The part of the string on the left of x exerts a force T on the right part. Since power is given by the dot product of force and velocity and in the present case the (particle) velocity is in the y-direction, we need to multiply the y component of the tension with the y component of the particle's velocity to get the power. So the (instantaneous) power transmitted by the left part to the right is

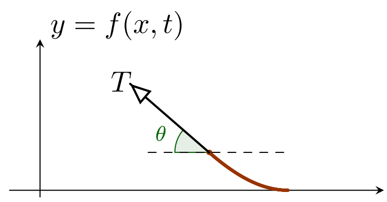


Figure 5: The left part of the string acts with tension T on the right part.

$$P(x,t) = (T\sin\theta)\, v_y$$

Since the angle $\theta \ll 1$, $\sin\theta \approx \tan\theta$ and $v_y = \partial y/\partial t$, so we get

$$P(x,t) \approx T\tan\theta\frac{\partial y}{\partial t} = -T\tan(\pi - \theta)\frac{\partial y}{\partial t}$$

But $\tan(\pi - \theta)$ is the slope of the tangent at x which is $\partial y/\partial x$. Hence, the instantaneous power

$$P(x,t) = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \tag{2.14}$$

For the case of harmonic wave

$$y = A\cos(kx - \omega t + \delta)$$

we get

$$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t + \delta), \quad \text{and } \frac{\partial y}{\partial t} = A\omega\sin(kx - \omega t + \delta)$$

Plugging these into Equation 2.14, we get the instantaneous power being transmitted across a point located at x as

$$P(x,t) = TA^2k\omega\sin^2(kx - \omega t + \delta)$$

The instantaneous power is thus *not* constant but oscillates with a frequency that is twice the frequency of the wave. It is not constant because the power input is oscillating. Usually, in such situations, the average value is taken over a complete cycle. Since the average of $\sin^2\theta$ (or $\cos^2\theta$) in a cycle is just $\frac{1}{2}$, we get the expression for the (average) power P transmitted by the wave across any point is

$$P = \frac{1}{2}TA^2k\omega = \frac{TA^2\omega^2}{2v} \tag{2.15}$$

which interestingly does not depend on x. This means that the wave is transmitting the same power through every point of the string. The fact that *the rate of transfer of energy depends on the square of the wave amplitude and the square of the frequency* is true in general, holding for all types of waves.

In a three-dimensional wave such as light or sound wave emitted from a point source, it is more significant to speak of the *intensity* of the wave, which is defined as *the power transmitted across a unit area normal to the direction in which the wave is travelling.* Just as with the power, the intensity is always proportional to the square of the amplitude. As shown in Figure 6, suppose

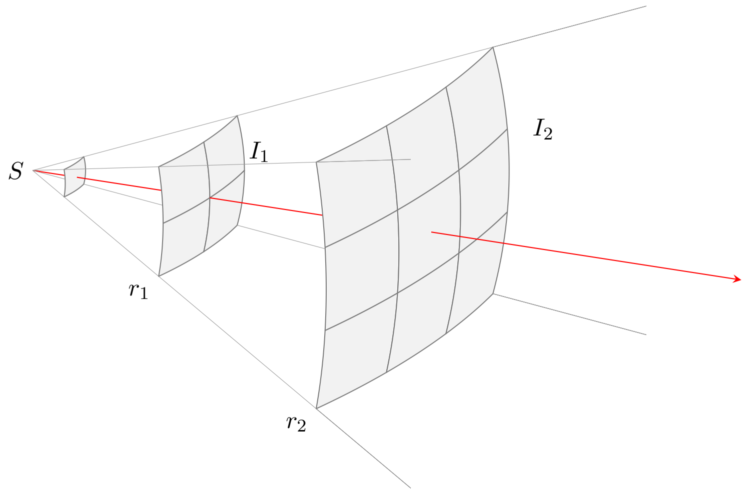


Figure 6: Spherical waves travelling outwards from a point source S.

spherical wave spread outward radially from a point source S generating power P. As the wave expands from a distance r_1 from the source to a distance r_2, its surface area increases from a value of $4\pi r_1^2$ to $4\pi r_2^2$. So if the intensity at these two radii be I_1 and I_2 respectively, then

$$I_1 = \frac{P}{4\pi r_1^2}, \quad \text{and } I_2 = \frac{P}{4\pi r_2^2}$$

which gives us

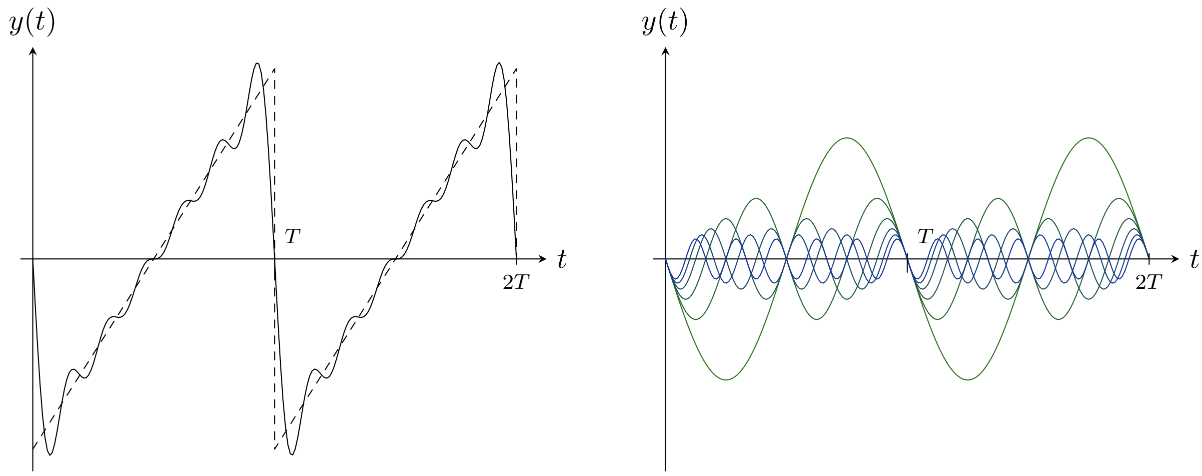
$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \tag{2.16}$$

Hence, the *intensity falls as the square of the distance from the source.* Since the intensity is proportional to the square of the amplitude, the *amplitude of a spherical wave varies inversely* *as the distance from the source*, so that a harmonic spherical wave moving radially out with wave number k and frequency ω can be written as

$$y = \frac{A}{r}\cos(kr - \omega t + \delta) \tag{2.17}$$

3 The Superposition Principle

It is an experimental fact that for many kinds of waves *two or more waves can traverse the same space independently of one another.* The fact that waves act independently of one another means that the displacement of any particle at a given time is simply the sum of the displacements that the individual waves alone would give it. This process of vector addition of displacements of a particle is called *superposition.* For example, radio waves of many frequencies pass through a radio antenna; the electric currents set up in the antenna by the superposed action of all these waves are very complex. Nevertheless, we can still tune to a particular station, the signal that we receive from it being in principle the same as that which we would receive if all other stations were to stop broadcasting.
For waves in deformable media *the superposition principle holds whenever the mathematical relation between the deformation and the restoring force is one of simple proportionality.* That is, the equation relating the restoring force and deformation is *linear.* In case, the relationship is not linear, the principle of superposition fails. Physically this happens when the wave disturbance is relatively large. For example, violent explosion creates shock waves in air. Although shock waves are longitudinal elastic waves in air, they behave differently from ordinary sound waves and the principle of superposition fails because the equations governing them are quadratic. The



(a) (b)

Figure 7: (a) The dashed line in a "sawtooth wave" commonly encountered in electronics. The Fourier series for this function is $y(t) = -\sin\omega t\frac{1}{2}\sin 2\omega t - \frac{1}{3}\sin 3\omega t - \ldots$. The solid line is the sum of the first six terms of this series and can be seen to approximate the sawtooth quite closely, except for overshooting at the corners. As more terms are included, the approximation gets better and better. (b) Here are shown the first six terms of the Fourier series which, when added, yield the solid curve of part (a).

importance of the superposition principle physically is that, where it holds, it makes it possible to analyse a complicated wave motion as a combination of simple waves. In fact, as was shown by the French mathematician *Jean-Baptiste Joseph Fourier*[3] (1768–1830), all that we need to build up a general period wave are simple harmonic waves. In a theorem that goes by his name,

[3] He is also credited for the discovery of greenhouse effect.

he showed that if $y(t)$ is any periodic motion of the source that generates the wave having period T, then

$$\begin{aligned} y(t) = A_0 + A_1 \cos \omega t \ &+ A_2 \cos 2\omega t + A_3 \cos 3\omega t + \ldots \\ &+ B_1 \sin \omega t + B_2 \sin 2\omega t + B_3 \sin 3\omega t + \ldots \end{aligned} \tag{3.1}$$

where $\omega = 2\pi/T$. The A's and B's are constants that have definite values for any particular periodic motion $y(t)$. (See Figure 7, for example.) If the motion is not periodic, as a pulse, the sum is replaced by an integral, the so called Fourier integral. Hence, any motion of the source of waves can be represented in terms of simple harmonic oscillations. Since the motion of the source creates the waves, it is quite clear that the waves themselves can be analysed as combinations of simple harmonic waves. Herein lies the importance of simple harmonic motion and simple harmonic waves.

3.1 Interference of Waves

Interference refers to the physical effects of superimposing two or more wave disturbances. According to the principle of superposition, if y_1 is the displacement of the string caused by the first wave and y_2 the corresponding quantity for the second then the total displacement of the string due to both the waves is $y = y_1 + y_2$.
Let us first consider that the two waves – both moving along the x axis – have the same amplitude A and frequency ω but differ in phase by an angle δ. More specifically, let the two waves be represented as

$$y_1 = A\cos(kx - \omega t) \tag{3.2}$$
$$y_2 = A\cos(kx - \omega t - \delta) \tag{3.3}$$

We have assumed that the second one *lags* in phase from the first one by an amount δ. The resultant disturbance, is therefore

$$y = y_1 + y_2 = A(\cos(kx - \omega t) + \cos(kx - \omega t - \delta))$$

Using the trigonometric identity $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$, the resultant can be written as

$$y = \left(2A\cos\frac{\delta}{2}\right)\cos\left(kx - \omega t - \frac{\delta}{2}\right) \tag{3.4}$$

We see that the resulting wave is also a simple harmonic wave having the same frequency. However, *the amplitude* $2A\cos(\delta/2)$ *of the resultant depends on the phase difference between the two waves.* When δ is *zero*, the two waves have the same phase everywhere. The crest of one corresponds to the crest of the other and likewise for the troughs. The resulting wave has an amplitude of $2A$ in this case and the waves are said to interfere *constructively*. On the other hand, when $\delta = 180°$, $\cos(\delta/2) = 0$ and so the amplitude is zero. In this case, the crest of one wave corresponds exactly to the troughs of the other and the waves are said to interfere *destructively*. In Figure 8a, we show the superposition of two waves (thin lines) almost in phase (δ is very small) and in Figure 8b the superposition of two waves almost 180° out of phase. In each case the resulting displacement (thick line) at any point is obtained by adding algebraically the ordinates of the component waves.
If the two harmonic waves are of equal frequency but unequal amplitudes, their superposition will still yield a harmonic wave. Let the two waves be given by

$$y_1 = A\cos(kx - \omega t) \tag{3.5}$$
$$y_2 = B\cos(kx - \omega t - \delta) \tag{3.6}$$

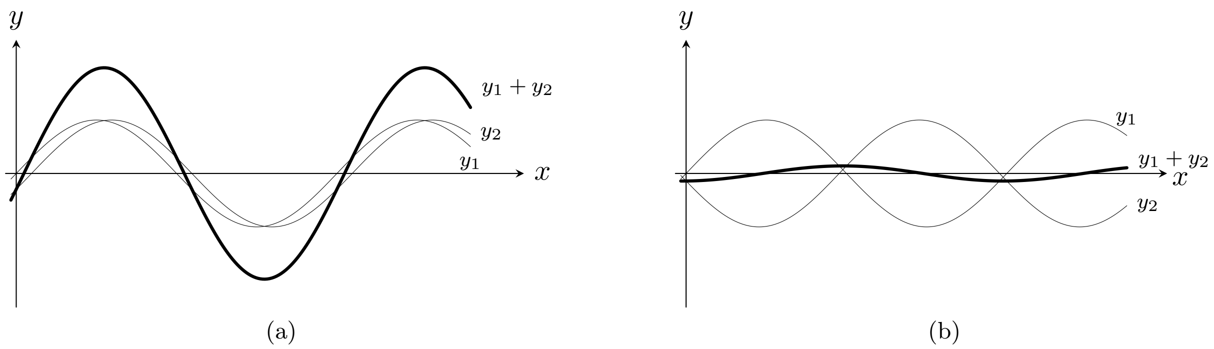
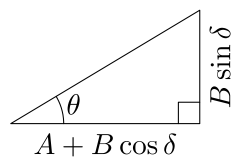


Figure 8: (a) The superposition of two waves of equal frequency and amplitude that are almost *in* phase results in a wave of almost twice the amplitude of either component. (b) The superposition of two waves of equal frequency and amplitude that are almost *out of* phase results in a wave whose amplitude is nearly zero. Notice that in both cases the frequency is unchanged. (The drawings are for $t = 0$.)

Then their sum



$$\begin{aligned} y &= y_1 + y_2 \\ &= A\cos(kx - \omega t) + B\cos(kx - \omega t - \delta) \\ &= A\cos(kx - \omega t) + B\cos(kx - \omega t)\cos\delta + B\sin(kx - \omega t)\sin\delta \\ &= (A + B\cos\delta)\cos(kx - \omega t) + B\sin\delta\sin(kx - \omega t) \end{aligned}$$

Now consider a right triangle, as shown above, whose altitude is $B\sin\delta$ and the base is $A + B\cos\delta$, so that the hypotenuse is

$$\begin{aligned} A' &= \sqrt{(A + B\cos\delta)^2 + B^2\sin^2\delta} \\ \Rightarrow A' &= \sqrt{A^2 + B^2 + 2AB\cos\delta} \end{aligned}$$

In terms of the angle θ, we can write $A + B\cos\delta = A'\cos\theta$ and $B\sin\delta = A'\sin\theta$, so that the resulting wave can be written as

$$y = A'\cos(kx - \omega t - \theta), \quad \left(A' = \sqrt{A^2 + B^2 + 2AB\cos\delta},\ \tan\theta = \frac{B\sin\delta}{A + B\cos\delta}\right) \tag{3.7}$$

which is a harmonic wave again.

On the other hand, if two harmonic waves of different frequencies are added the resulting wave is no longer a harmonic wave. The superposition principle is still valid; however the resulting wave is *complex* and has no fixed relations to its components. For example, in Figures 9a and 9b, we have added two waves travelling along the x-axis having the same amplitude but having frequencies in the ratio 3:1; the phase relation is changed from Figure 9a to 9b and we see how changing the phase relation may produce a resultant of very different form.

In Figure 10, three waves of different frequencies, and amplitudes are added. The resultant complex wave is quite different from a simple periodic wave, and in this respect resembles waveforms normally generated by musical instruments. In Figure 11 a wave of very high frequency is added to one of very low frequency. Each component frequency is clearly discernible in the resultant. In all these cases, as the component waves travel with the the same velocity, the resultant waveform moves with the same velocity and the wave shape remains unchanged.

3.2 Standing Waves

So far we considered superposition of waves travelling in the same direction. Let us consider now the situation where waves travelling in *opposite* directions are added. Such a situation

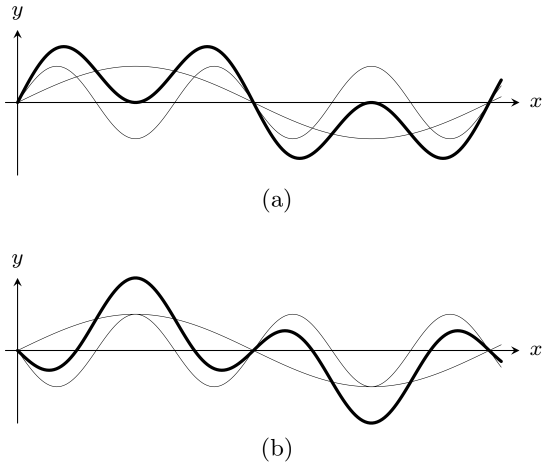


Figure 9: The addition of two waves with frequency ratio 3:1 (thin lines) yields a wave whose shape (thick line) depends on the phase relationship of the components. Compare (a) and (b).

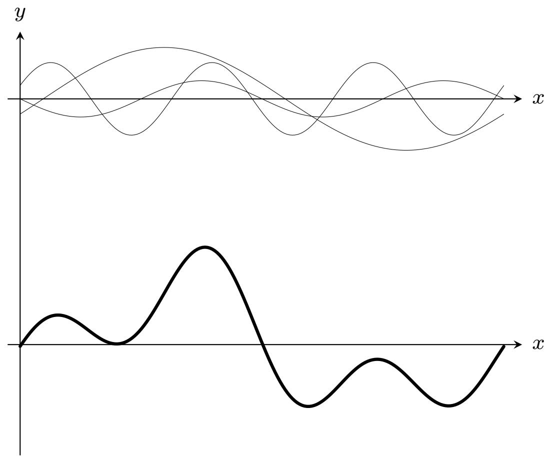


Figure 10: The addition of three waves (top) of different frequencies yields a complex waveform (bottom).

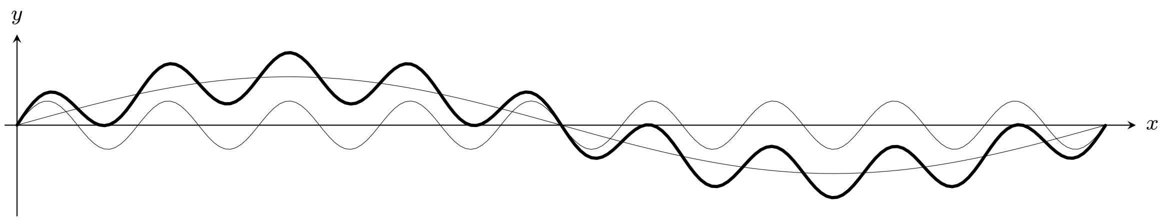


Figure 11: The addition (thick line) of two harmonic waves of widely different frequencies (thin lines).

arises when, for example, waves are incident on a boundary separating two different media. In such cases, part of the wave gets *transmitted* to the other medium while the rest of the wave gets *reflected* travelling in the same medium (in which the incident wave is moving) but in the opposite direction. If the wave gets reflected from a rigid wall, there is no transmitted wave while the entire wave is reflected back with a phase shift of π. In this case, the string will have two waves moving in opposite directions with a phase difference of π. Let the two (harmonic) waves be given by

$$y_1 = A\cos(kx - \omega t) \tag{3.8}$$
$$y_2 = A\cos(-kx - \omega t + \pi) = -A\cos(kx + \omega t) \tag{3.9}$$

Hence, the displacement of any point is the sum of these:

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\cos(kx - \omega t) - A\cos(kx + \omega t) \\ \Rightarrow y &= 2A\sin kx \sin \omega t \end{aligned} \tag{3.10}$$

We see that this equation no longer represents a travelling wave; it represents a *standing wave.* Notice that a particle located at x simply performs simple harmonic motion with an amplitude $2A\sin kx$. Indeed this feature distinguishes a standing wave from a travelling wave: *In a standing wave, the amplitude is not the same for all particles but varies with the location of the particle* while in a travelling wave all the particles of the medium oscillate with the same amplitude[4]. Also

[4]We are speaking strictly in context of plane harmonic waves.

notice that all the particles are oscillating in phase. As the amplitude depends on the location x of the particle, we can locate certain points along the string whose amplitude is zero; they are the points at which $kx = 0$ which gives

$$kx = n\pi \quad \Rightarrow \frac{2\pi}{\lambda}x = n\pi \quad \Rightarrow x = \frac{n\lambda}{2} \quad \text{for } n = 1, 2, 3, \ldots$$

that is, $x = \lambda/2,\ \lambda,\ 3\lambda/2,\ 2\lambda$, etc. These points are called *nodes* and are spaced one-half wavelengths apart. Similarly, there are points where the amplitude has the maximum value of $2A$; this happens at points for which

$$kx = (2n-1)\frac{\pi}{2} \quad \Rightarrow \frac{2\pi}{\lambda}x = (2n-1)n\frac{\pi}{2} \quad \Rightarrow x = (2n-1)\frac{\lambda}{4} \quad \text{for } n = 1, 2, 3, \ldots$$

that is, $x = \lambda/4,\ 3\lambda/4,\ 5\lambda/4$, etc. These points are called *antinodes* and are spaced one-half wavelengths apart. The separation between a node and an adjacent antinode is one-quarter wavelength.
It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodal points in the string which are always at rest. Hence, the energy remains "standing" in the string, although it does alternates between the oscillatory kinetic energy and the string potential energy. We call it a wave because it results from the superposition of waves travelling in opposite directions. Figure 12 (a), (b), (c), (d) shows a standing wave pattern

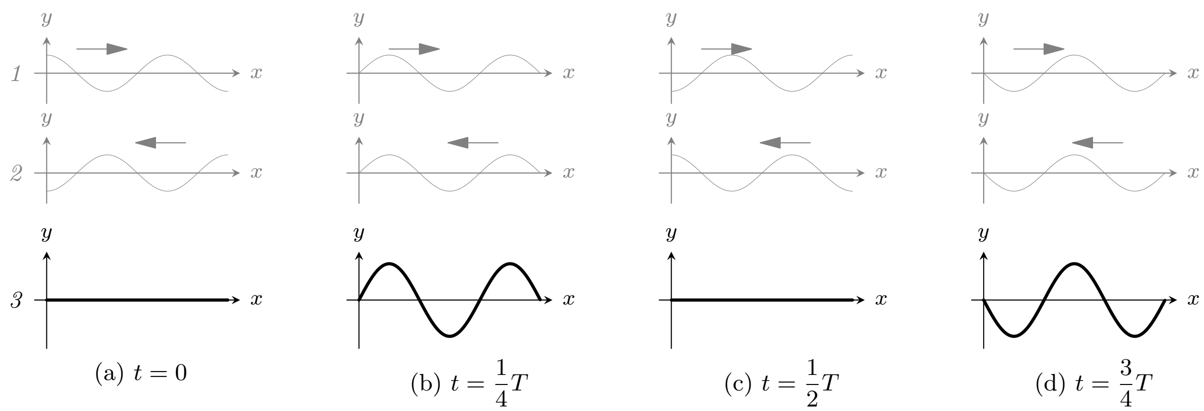


Figure 12: Standing waves as the superposition of right and left going waves; *1* and *2* are the components, *3* is the resultant.

separately at intervals of one-quarter of a period in the lower figures, in the row marked *3*. The travelling waves, one moving along the positive x-axis (row *1*) and the other along the negative x-axis (row *2*), whose superposition give rise to the standing wave, are also shown for the same quarter-period intervals. Usually, oscillating strings vibrate so rapidly that the eye perceives only

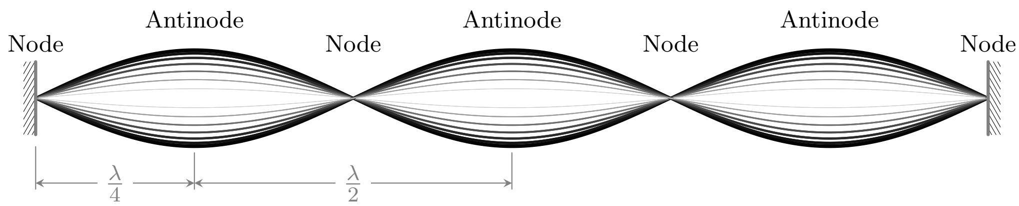


Figure 13: The envelope of a standing wave, corresponding to a time exposure of the motion, and showing the patterns of nodes and antinodes.

a blur whose shape is that of the envelope of the motion, an example of which has been shown in Figure 13.

3.3 Beats

Earlier we have seen that when two harmonic waves of different frequencies are added the result is a complex wave and no longer a harmonic. However, when their frequencies are only *slightly* different, their superposition gives rise to the phenomenon of *beats*.
Suppose two harmonic waves pass through some point P along the x-axis. One of the waves has angular frequency ω_1 and the other ω_2 such that $|\omega_1 - \omega_2| \ll \omega_1$ or ω_2. As the waves pass by P, they impart P simple harmonic oscillations which has been plotted as a function of time in Figure 14(a). The superposition of these oscillations determines the resultant displacement of P. Let the individual oscillations imparted by these two waves to P be given by

$$y_1 = A\cos\omega_1 t$$
$$\text{and } y_2 = A\cos\omega_2 t$$

For simplicity we have assumed the amplitudes of these oscillations to be same although this is not necessary. By the superposition principle, the total displacement of the point P is

$$y = y_1 + y_2 = A(\cos\omega_1 t + \cos\omega_2 t) = 2A\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

which we regroup as

$$y = \left\{2A\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\right\}\cos\left(\frac{\omega_1+\omega_2}{2}t\right) \tag{3.11}$$

The resulting vibration has been shown in Figure 14(b). It may be considered to have a fre-

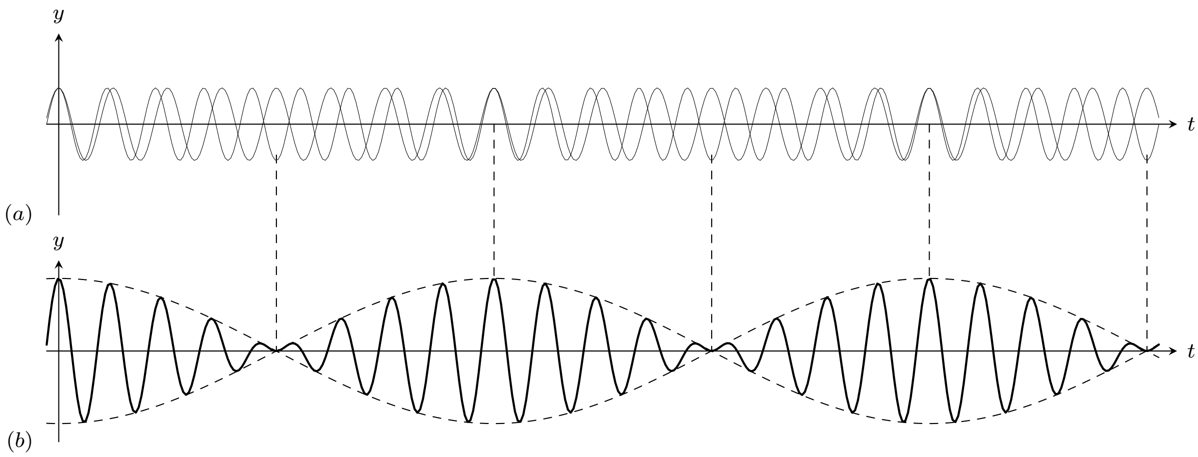


Figure 14: The beat phenomenon. Two waves of slightly different frequencies, shown in (a), combine to give a wave whose amplitude (dashed line) varies periodically with time.

quency $\dfrac{\omega_1+\omega_2}{2}$, which is the average frequency of the two waves, and an amplitude given by the expression inside the braces. Hence, *the amplitude varies slowly* with time with a frequency $\omega_{\text{amp}} = \dfrac{\omega_1-\omega_2}{2}$ which is quite small as ω_1 and ω_2 are nearly equal.
A beat, that is, *a maximum of the amplitude*, will occur when the term $\cos\left(\dfrac{\omega_1-\omega_2}{2}t\right)$ is either 1 or -1. Since *each* of these values are achieved once in each cycle (Figure 14(b)), the number of beats per second is *twice* the frequency ω_{amp}. Hence, the (angular) frequency of beats is

$$\Omega_b = |\omega_1 - \omega_2| \tag{3.12}$$

that is to say that the beat frequency is simply the difference of the frequencies of the component waves. Beats between two *tones* in sound waves can be detected by human ear up to a frequency of about 7 per second.