Fluid Mechanics

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Contents

1 Introduction

Matter is normally classified as being in one of three states: solid, liquid, or gas. From everyday experience, we know that a solid has a definite volume and shape. A brick maintains its familiar shape and size day in and day out. We also know that a liquid has a definite volume but no definite shape. Finally, we know that an unconfined gas has neither a definite volume nor a definite shape. These definitions help us picture the states of matter, but they are somewhat artificial. For example, asphalt and plastics are normally considered solids, but over long periods of time they tend to flow like liquids. Likewise, most substances can be a solid, a liquid, or a gas (or a combination of any of these), depending on the temperature and pressure. In general, the time it takes a particular substance to change its shape in response to an external force determines whether we treat the substance as a solid, as a liquid, or as a gas. A fluid is a collection of molecules that are randomly arranged and held together by weak cohesive forces and by forces exerted by the walls of a container. Both liquids and gases are fluids.

The main property that distinguishes a fluid from a solid is that *a fluid cannot* **maintain** *a shear stress (i.e. a force applied parallel to the surface), for any length of time.* If a shear is applied to a fluid, it will move under the shear. Thicker liquids like honey move less easily than fluids like air or water. The measure of the ease with which a fluid yields is its *viscosity*, which we shall take up a bit later; for the moment we consider only those situations in which viscosity can be ignored.

2 Hydrostatics

We begin by considering *hydrostatics*, the theory of liquids at rest. When liquids are at rest, there are no shear forces (even for viscous liquids). The law of hydrostatics, therefore, is that **the forces are always normal to *any* surface inside the fluid.**

▷ **Definition 1** (**Pressure**)**.** The normal force per unit area is called pressure.

From the fact that there is no shear in a static fluid it follows that the pressure at any point is same in all directions (see Fig. 1).
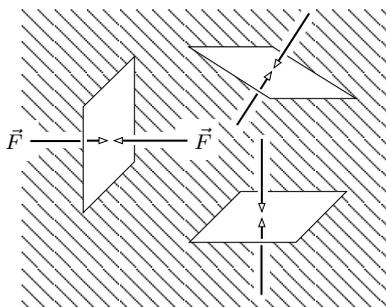

Figure 1: In a static fluid the force per unit area across any surface is normal to the surface and is same for all orientation of the surface.
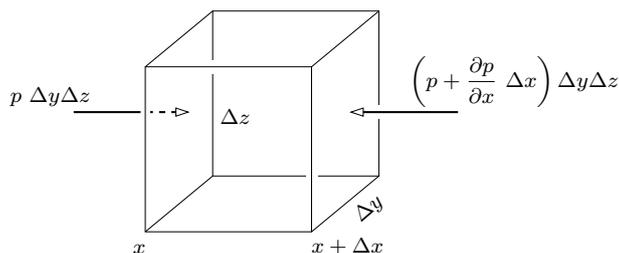

Figure 2: Finding the pressure force on a cube.

The pressure in a fluid may vary from place to place. For example, in a static fluid at the Earth's surface the pressure will vary with height because of the weight of the fluid. In the following, we derive a very useful relation.

If we take a small cube of the fluid in the fluid itself(!), what is the net force on it because of the pressure? Since the pressure at any point is same in all directions, there can be a net force per unit volume only because the pressure varies from one point to another. Suppose that the pressure varies in the x-direction — and we take the coordinate directions parallel to the cube

edges. The pressure on the face at x gives the force $p \, \Delta y \Delta z$, and the pressure on the face at $x + \Delta x$ gives the force $-\left(p + \dfrac{\partial p}{\partial x} \Delta x \right) \Delta y \Delta z$. (The expression $\partial p / \partial x$ means "the rate of change of p along the x axis only, treating y and z as constants". So that $(\partial p / \partial x)\Delta x$ is the change in pressure as one goes from x to $x + \Delta x$.) Thus the resultant force on the cube along the x-axis is $-(\partial p / \partial x) \, \Delta x \Delta y \Delta z$. If we take the remaining pairs of faces of the cube, it is see that the resultant pressure force on the cube is

$$ -\left(\frac{\partial p}{\partial x} \, \widehat{\mathbf{i}} + \frac{\partial p}{\partial y} \, \widehat{\mathbf{j}} + \frac{\partial p}{\partial z} \, \widehat{\mathbf{k}} \right) \Delta x \Delta y \Delta z $$

The quantity in the parentheses is simply written as ∇p, (read: "gradient of p") where the symbol "∇" means the symbolic "*operator*"

$$ \frac{\partial}{\partial x} \, \widehat{\mathbf{i}} + \frac{\partial}{\partial y} \, \widehat{\mathbf{j}} + \frac{\partial}{\partial z} \, \widehat{\mathbf{k}} $$

Thus, the *pressure force per unit volume* on the cube is $-\nabla p$. If there are other forces in addition — such as gravity — then the pressure must balance them to give equilibrium.

Let's take a circumstance in which such an additional force can be described by a potential energy, as would be true in the case of gravitation; we will let ϕ stand for the potential energy per unit mass. (For gravity, for instance, ϕ is just gz.) The force per unit mass is given in terms of the potential energy $-\nabla \phi$, and if ρ is the density of the fluid, the force per unit volume is $-\rho \nabla \phi$. For equilibrium this force per unit volume added to the pressure force per unit volume must give zero:

$$ -\nabla p - \rho \nabla \phi = 0 \tag{2.1} $$

Eq. (2.1) is the equation of hydrostatics. In *general*, it **has no solution**. If the density varies in space in an arbitrary way, there is no way for the forces to be in balance, and the fluid cannot be in static equilibrium. Convection currents will start up. We can see this from the equation since the pressure term is a pure gradient, whereas for variable ϕ, the other term is not. Only when ρ is a constant (**incompressible fluid**) is the potential term a pure gradient. Since the density ρ is a constant, we could write Eq. (2.1) as

$$ -\nabla p - \nabla(\rho \, \phi) = 0 \qquad \Rightarrow \qquad -\nabla(p + \rho \, \phi) = 0 $$

But that means, the solution is

$$ p + \rho \, \phi = 0 \tag{2.2} $$

Another possibility which allows hydrostatic equilibrium is for ρ to be a function only of p. However, we shall not consider that situation.

As a special cases of Eq. (2.2), we find the following two useful results:

- $\phi = $ constant: If the field potential is constant which means that the force field is absent, then the pressure p is same throughout the liquid column. This is the ***Pascal's law***.

- If a liquid column is kept in the uniform field of gravity as shown in Fig. 3, then $\phi = gz$. Taking the zero of the potential energy at the surface of the liquid and taking the pressure as the atmospheric pressure p_0, we get

$$ p_0 = p - \rho g h $$
$$ \Rightarrow \qquad p = p_0 + \rho g h \tag{2.3} $$

that is ***the pressure p at a depth h below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount $\rho g h$.***

This can be seen in another way: consider the column of liquid with its free body diagram show in Fig. 3. Equating the forces on a cross section A, we obtain

$$p_0 A + (\rho A h)g = pA \qquad \Rightarrow \qquad p = p_0 + \rho g h$$
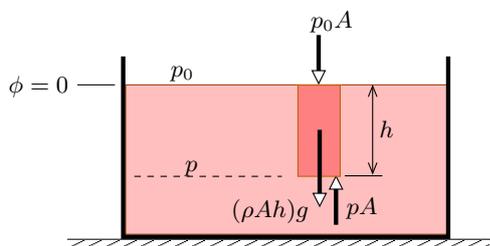
which is the same equation as (2.3) above.

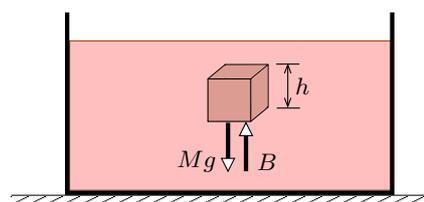Figure 3: Dependence of pressure on height of a liquid column.

Figure 4: Origin of buoyant force.

2.1 Buoyant forces and Archimedes's Principle

The upward force exerted by a fluid on any immersed object is called a *buoyant* force. Of course, this is applicable only in the uniform gravitational field of Earth.

We can determine the magnitude of a buoyant force exerted by water by applying some logic and Newton's second law. Imagine that, instead of air, a beach ball is filled with water. If you were standing on land, it would be difficult to hold the water-filled ball in your arms. If you held the ball while standing neck deep in a pool, however, the force you would need to hold it would almost disappear. In fact, the required force would be zero if we were to ignore the thin layer of plastic of which the beach ball is made. Because the water-filled ball is in equilibrium while it is submerged, the magnitude of the upward buoyant force must equal its weight.

If the submerged ball were filled with air rather than water, then the upward buoyant force exerted by the surrounding water would still be present. However, because the weight of the water is now replaced by the much smaller weight of that volume of air, the net force is upward and quite great; as a result, the ball is pushed to the surface.

The manner in which buoyant forces act is summarized by *Archimedes's principle*, which states that **the magnitude of the buoyant force always equals the weight of the fluid displaced by the object.** The buoyant force acts vertically upward through the point that was the center of gravity of the displaced fluid.

Note that Archimedes's principle does not refer to the makeup of the object experiencing the buoyant force. The object's composition is not a factor in the buoyant force.

Although we have described the magnitude and direction of the buoyant force, we still do not know its origin. Why would a fluid exert such a strange force, almost as if the fluid were trying to expel a foreign body? To understand why, look at Fig. 4. The pressure at the bottom of the cube is greater than the pressure at the top by an amount $\rho g h$, where h is the length of any side of the cube. The pressure difference Δp between the bottom and top faces of the cube is equal to the buoyant force per unit area of those facesthat is, $\Delta p = B/A$. Therefore, $B = (\Delta p)A = \rho g h A = \rho g V$, where V is the volume of the cube. Because the mass of the fluid in the cube is $M = \rho V$, we see that

$$B = \rho g V = Mg$$

where Mg is the weight of the fluid in the cube. Thus, the buoyant force is a result of the pressure differential on a submerged or partly submerged object.

3 Hydrodynamics

Thus far, our study of fluids has been restricted to fluids at rest. We now turn our attention to fluids in motion. Instead of trying to study the motion of each particle of the fluid as a function of time, we describe the properties of a moving fluid at each point as a function of time.

3.1 Flow characteristics

When fluid is in motion, its flow can be characterized as being one of two main types. The flow is said to be **steady**, or **laminar**, if each particle of the fluid follows a smooth path, such that the paths of different particles never cross each other, as shown in Fig. 5. In steady flow, the velocity of the fluid at any point remains constant in time.

Figure 5: Laminar flow around an automobile in a test wind tunnel.

Above a certain critical speed, fluid flow becomes **turbulent**; turbulent flow is irregular flow characterized by small whirlpool-like regions.

The term **viscosity** is commonly used in the description of fluid flow to characterize the degree of internal friction in the fluid. This internal friction, or *viscous force*, is associated with the resistance that two adjacent layers of fluid have to moving relative to each other. Viscosity causes part of the kinetic energy of a fluid to be converted to internal energy. This mechanism is similar to the one by which an object sliding on a rough horizontal surface loses kinetic energy.

Because the motion of real fluids is very complex and not fully understood, we make some simplifying assumptions in our approach. In our model of an **ideal fluid**, we make the following four assumptions:

(i) *The fluid is non-viscous.* In a non-viscous fluid, internal friction is neglected. An object moving through the fluid experiences no viscous force.

(ii) *The flow is steady.* In steady (laminar) flow, the velocity of the fluid at each point remains constant.

(iii) *The fluid is incompressible.* The density of an incompressible fluid is constant.

(iv) *The flow is irrotational.* In irrotational flow, the fluid has no angular momentum about any point. If a small paddle wheel placed anywhere in the fluid does not rotate about the wheels center of mass, then the flow is irrotational.

3.2   Streamlines and Equation of Continuity

If the velocity of the fluid at a given point in space is the same at all times (such flow is said to be "steady"), we can draw curves the tangent of which at any point is the direction of the fluid velocity there. These curves are called **streamlines**.

For example, Fig. 6 shows a streamline from P to Q. The curve describes the trajectory of a fluid particle in time. The trajectory remains the same no matter what the initial or starting time; the curve PQ is like a permanent map of fluid flow, indicating how the fluid streams. A set of streamlines like the ones shown in Fig. 7, form a *tube of flow*.
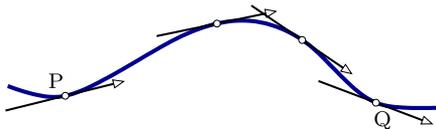

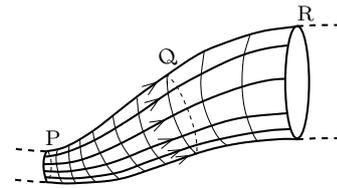Figure 6: Streamline: the trajectory of a particle.

Figure 7: A tube of flow.

Clearly, *two streamlines do not intersect*, because then the fluid would have two different velocities at one point. How close to each other do we draw streamlines? If we intend to show every flowing particle, or every trajectory, we would end up with a continuum of lines. So, what is a good way of spacing them? Consider planes perpendicular to the direction of fluid flow, *e.g.* at three points P, Q and R (Fig. 7). The plane pieces are so chosen that their boundaries are determined by the same set of streamlines. This means that the number per second of fluid particles 'inside', *i.e.* crossing the surfaces indicated at P, Q and R is the same. (This is called the *condition of continuity.*) Thus, if the magnitudes of of velocities at P, Q and R are v_P, v_Q and v_R, then the areas A_P, A_Q and A_R of the planes must be such that

$$A_\mathrm{P}\, v_\mathrm{P} = A_\mathrm{Q}\, v_\mathrm{Q} = A_\mathrm{R}\, v_\mathrm{R} \tag{3.1}$$

This suggests that we space streamlines inversely with the *magnitudes* of the velocities at P, Q and R. Then, by looking at the number of streamlines perpendicular to the direction of fluid flow, we can estimate the fluid velocity. If the streamlines are close, the velocity is large, and vice versa.

3.3   Bernoulli's theorem

The theorem of Bernoulli is in fact nothing more than a statement of the conservation of energy. A conservation theorem such as this gives us a lot of information about a flow without our actually having to solve the detailed equations.
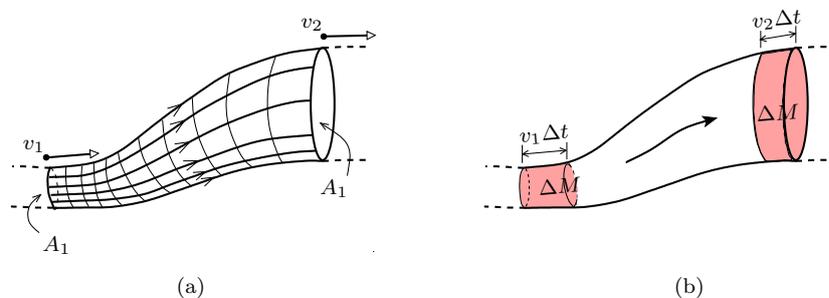

Figure 8: Fluid motion in a flow tube.

Imagine a bundle of adjacent streamlines which form a tube of flow as shown in Fig. 8(a). Since the walls of the tube consists of streamlines, no fluid flows out through the wall. Let's call

the area at one end of the stream tube A_1, the fluid speed there v_1, the density of the fluid ρ_1, and the potential energy ϕ_1. At the other end of the tube, we have the corresponding quantities A_2, v_2, ρ_2, and ϕ_2.

After a short interval of time Δt, the fluid at A_1 has moved a distance $v_1\Delta t$, and the fluid at A_2 has moved a distance $v_2\Delta t$ (Fig. 8(b)). The conservation of *mass* requires that the mass which enters through A_1 must be equal to the mass which leaves through A_2. These masses at these two ends must be the same:

$$\Delta M = \rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$$

So we have the equality

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \tag{3.2}$$

This equation tells us that the velocity varies inversely with the area of the flow tube if ρ is constant.

Now we calculate the work done by the fluid pressure. The work done on the fluid entering at A_1 is $p_1 A_1 v_1 \Delta t$, and the work given up at A_2 is $p_2 A_2 v_2 \Delta t$. The net work on the fluid between A_1 and A_2 is, therefore,

$$p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t$$

which must equal the increase in the energy of a mass ΔM of fluid in going from A_1 to A_2. In other words,

$$p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t = \Delta M (E_2 - E_1), \tag{3.3}$$

where E_1 is the energy per unit mass of fluid at A_1, and E_2 is the energy per unit mass at A_2. The energy per unit mass of the fluid can be written as

$$E = \frac{1}{2} v^2 + \phi + U,$$

where $\frac{1}{2} v^2$ is the kinetic energy per unit mass, ϕ is the potential energy per unit mass, and U is an additional term which represents the internal energy per unit mass of fluid. The internal energy might correspond, for example, to the thermal energy in a compressible fluid, or to chemical energy. All these quantities can vary from point to point. Using this form for the energies in Eq. (3.3), we have

$$\frac{p_1 A_1 v_1 \Delta t}{\Delta M} - \frac{p_2 A_2 v_2 \Delta t}{\Delta M} = \frac{1}{2} v_2^2 + \phi_2 + U_2 - \frac{1}{2} v_1^2 - \phi_1 - U_1$$

But $\Delta M = \rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$, so we get

$$\frac{p_1}{\rho_1} + \frac{1}{2} v_1^2 + \phi_1 + U_1 = \frac{p_2}{\rho_2} + \frac{1}{2} v_2^2 + \phi_2 + U_2 \tag{3.4}$$

This result is the Bernoulli's theorem. If the fluid is incompressible, and the internal energy term is same on both sides, then multiplying throughout by ρ, we get

$$p + \frac{1}{2}\rho v^2 + \rho\phi = \text{constant} \tag{3.5}$$

along any streamline. Further, if ϕ is the gravitational potential energy per unit mass gh (where h is measured from a reference point), then we get the familiar form of the Bernoulli's equation:

$$p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \tag{3.6}$$

We consider now some simple examples in which the Bernoulli's equations gives a description gives a description of the flow. Suppose we have water flowing out of a hole near the bottom of a tank, as drawn in Fig. 9.
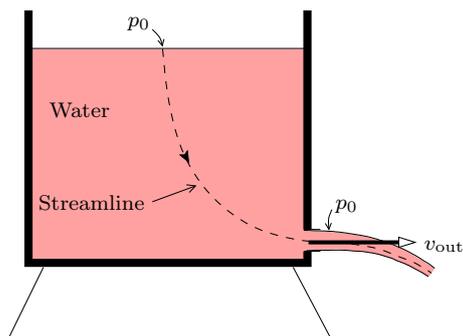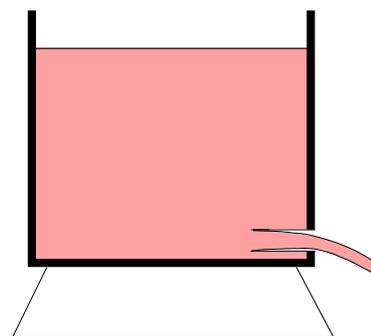
Figure 9: Flow from a tank.

Figure 10: With a re-entrant discharge tube, the stream contracts to one half the area of the opening.

We take a situation in which the flow speed v_{out} at the hole is much larger than the flow speed near the top of the tank; in other words, we imagine that the diameter of the tank is so large that we can neglect the drop in the liquid level. (We could make a more accurate calculation if we wished.) At the top of the tank the pressure is p_0, the atmospheric pressure, and the pressure at the sides of the jet is also p_0. Now we write our Bernoulli equation for a streamline, such as the one shown in the figure. At the top of the tank, we take v equal to zero and we also take the gravity potential energy to be zero. At the speed v_{out}, and $\phi = -gh$, so that

$$p_0 = p_0 + \frac{1}{2}\rho v_{\text{out}}^2 - \rho gh,$$

or

$$v_{\text{out}} = \sqrt{2gh} \tag{3.7}$$

This velocity is just what we would get for something which falls the distance h. It is not too surprising, since the water at the exit gains kinetic energy at the expense of the potential energy of the water at the top. Do not get the idea, however, that one can figure out the rate that the fluid flows out of the tank by multiplying this velocity by the area of the hole. The fluid velocities as the jet leaves the hole are not all parallel to each other but have components inward towards the center of the stream — the jet is converging. After the jet has gone a little way, the contraction stops and the velocities do become parallel. So the total flow is the velocity times the area *at that point*. In fact, if we have a discharge opening which is just a round hole with sharp edge, the jet contracts to 62 % of the area of the hole. The reduced effective area of the discharge varies for different shape of discharge tubes, and experimental contractions are available as tables of *efflux coefficients*.

If the discharge tube is re-entrant, as shown in Fig. 10, it is possible to prove that the efflux coefficient is exactly 50 %.

4 Viscosity

So far we have been speaking about ideal fluids (Feynman's "dry" water) — fluids which do not pose any resistance to flow; but real fluids, of course, pose friction.

We begin with an important experimental fact. It turns out – although it is *not* self evident at all – that in all circumstances where it has been experimentally checked, the *velocity of a fluid is (relatively) zero at the surface of a solid* that is in contact with the fluid. You have noticed, no doubt, that the blade of a fan will collect a thin layer of dust – and that it is still there after the fan has been churning up the air.

We originally characterized a liquid by the fact that if you put a shearing stress on it, no matter how small, it would give way. It flows. In static situations, there are no shearing stresses.

But before equilibrium is reached, there can be shearing forces. *Viscosity* describes these shear forces which exist in a moving fluid. To get a measure of the shear force during the motion of a fluid, we consider the following kind of experiment.
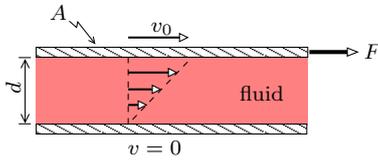

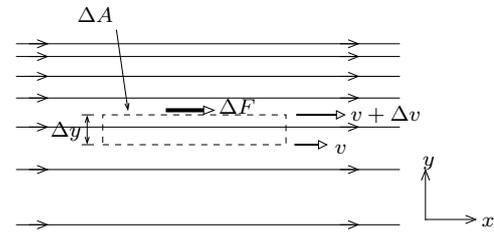Figure 11: Viscous drag between two parallel plates.

Figure 12: The shear stress in a viscous fluid.

Suppose that we have two solid plane surfaces with water between them, as in Fig. 11, and we keep one stationary while moving the other parallel to it at the slow speed v_0. If one measures the force required to keep the upper plate moving, it is found that it is proportional to the area of the plates and to v_0/d, where d is the distance between the plates. So the shear stress F/A is proportional to v_0/d:

$$\frac{F}{A} = \eta \, \frac{v_0}{d}.$$ (4.1)

The constant of proportionality η is called the *coefficient of viscosity.*

If we have a more complicated situation, we can always consider a little, flat, rectangular (actually a cuboidal) cell in water with its faces parallel to the flow, as in Fig. 12. The shear force across the cell is given by

$$\frac{\Delta F}{\Delta A} = \eta \, \frac{\Delta v_x}{\Delta y} = \eta \frac{\partial v_x}{\partial y}$$ (4.2)

In the more general case we write

$$\text{the shearing force in the } xy\text{-plane:} \quad S_{xy} = \eta \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)$$ (4.3)

There are of course the corresponding equations for S_{yz} and S_{zx}.

4.1 Velocity distribution in a fluid between rotating cylinders

As an example of application of these ideas, we consider the motion of a fluid between two coaxial cylinders. Let the inner one have the radius a and the angular speed ω_a, and let the outer one have radius b and speed ω_b. See Fig. 13.
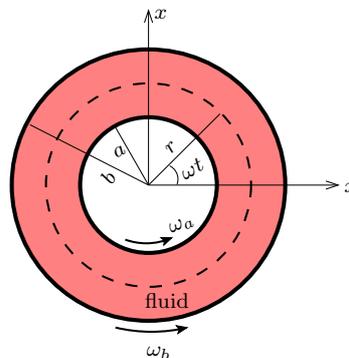

Figure 13: The flow of a fluid between two concentric cylinders rotating at different angular speed.

Our task is to determine the variation of the speed between the cylinders. To do so we begin by finding a formula for the viscous shear in the fluid at a distance r from the axis. From the symmetry of the problem, we can assume that the flow is always tangential and its speed depends only on r; $v = v(r)$. If we watch a "particle" of water at a distance r from the axis, its coordinates as a function of time are

$$x = r\cos\omega t, \qquad\qquad y = r\sin\omega t,$$

where $\omega = v/r$. Then the x- and y-components of the velocity are

$$v_x = -r\omega\sin\omega t = -\omega y \qquad \text{and} \qquad v_y = r\omega\cos\omega t = \omega x. \tag{4.4}$$

From Eq. (4.3), we have

$$S_{xy} = \eta\left[\frac{\partial}{\partial x}(x\omega) - \frac{\partial}{\partial y}(y\omega)\right] = \eta\left[x\frac{\partial\omega}{\partial x} - y\frac{\partial\omega}{\partial y}\right] \tag{4.5}$$

For a point at $y = 0$, $\partial\omega/\partial y = 0$, and $x\partial\omega/\partial x$ is the same as $r\,d\omega/dr$. So at that point

$$(S_{xy})_{y=0} = \eta r\frac{d\omega}{dr}. \tag{4.6}$$

The stress we have calculated is the tangential shear which is the same all around the cylinder at radius r. We can get the *torque* acting *across a cylindrical surface* at this radius by multiplying the shear stress by the moment arm r and the area $2\pi r\ell$. We get

$$\tau = 2\pi r^2\ell(S_{xy})_{y=0} = 2\pi\eta\ell r^3\frac{d\omega}{dr} \tag{4.7}$$

Since the motion of the water is steady–there is no angular acceleration–the net torque on the cylindrical shell of water between r and $r + dr$ must be zero; that is, the torque at r must be balanced by an equal and opposite torque at $r + dr$, so that τ must be independent of r. In other words, $r^3\,d\omega/dr$ must equal some constant, say A, and

$$\frac{d\omega}{dr} = \frac{A}{r^3} \tag{4.8}$$

Integrating, we find that ω varies with r as

$$\omega = -\frac{A}{2r^2} + B \tag{4.9}$$

The constants A and B are to be determined to fit the conditions that $\omega = \omega_a$ at $r = a$ and $\omega = \omega_b$ at $r = b$. Solving we get

$$A = \frac{2a^2b^2}{b^2 - a^2}(\omega_b - \omega_a), \qquad \text{and} \qquad B = \frac{b^2\omega_b - a^2\omega_a}{b^2 - a^2} \tag{4.10}$$

So we know ω as a function of r, and from it $v = \omega r$.

If we want the torque, we can get it from Eq. (4.7) and Eq. (4.8):

$$\tau = \frac{4\pi\eta\ell a^2b^2}{b^2 - a^2}(\omega_b - \omega_a) \tag{4.11}$$

It is proportional to the relative angular speeds of the two cylinders. One standard apparatus for measuring the coefficients of viscosity is built this way. One cylinder–say the outer one–is on pivots but is held stationary by a spring balance which measures the torque on it, while the inner one is rotated at constant angular speed. The coefficient of viscosity is then determined from Eq. (4.11).

From its definition, it can be seen that the SI units of η are newton-sec/m^2. For water at 20° C,

$$\eta = 10^{-3} \text{ newton-sec/m}^2$$

It is usually more convenient to use the *specific viscosity*, which is η divided by the density ρ. The values for water and air are then comparable:

$$\text{water at } 20° \text{ C:} \quad \eta/\rho = 10^{-6} \text{ m}^2/\text{s}$$
$$\text{air at } 20° \text{ C:} \quad \eta/\rho = 15 \times 10^{-6} \text{ m}^2/\text{s}$$

Viscosity usually depends strongly on temperature. For instance, for water just above the freezing point, η/ρ is 1.8 times larger than it is at 20° C.

4.2 Flow through a tube: Poiseiulle's law

As another example, we consider the flow of a viscous liquid across a horizontal right-circular cylindrical tube under a pressure difference of Δp across it. Let the length of the tube be ℓ and the radius be R (see Fig. 14).
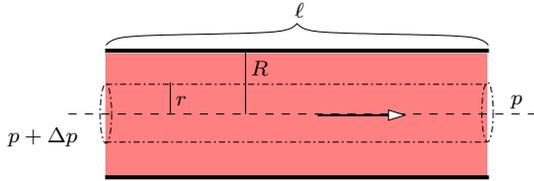

Figure 14: Flow of viscous fluid through a right circular cylindrical tube.
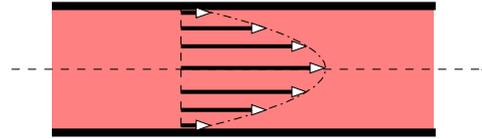

Figure 15: Velocity distribution of the fluid in the tube.

The pressure difference pushes the fluid out of the tube, but because of viscous forces, the velocity at all points in a cross-section is *not* the same. In fact, the fluid "particles" just in contact with the inner wall of the tube must be at rest. Thus, there is a velocity gradient as one moves away from the axis of the tube. Taking the radially away direction as positive, this gradient is $-\mathrm{d}v/\mathrm{d}r$.

If we take a coaxial cylinder at a radius r (Fig. 14), the pressure difference Δp exerts a force $\pi r^2 \Delta p$ on its cross section that pushes the fluid in this cylinder. This force acts across over the lateral surface area $2\pi r\ell$ of the cylinder at radius r. Thus, the shear stress is $\pi r^2 \Delta p/2\pi r\ell$. So we must have

$$\frac{\pi r^2 \Delta p}{2\pi r\ell} = -\eta \frac{\mathrm{d}v}{\mathrm{d}r} \qquad \Rightarrow \quad \mathrm{d}v = -\frac{\Delta p}{2\eta\ell} \, r\mathrm{d}r$$

Integrating this relation we obtain

$$v = -\frac{\Delta p}{2\eta\ell} \, \frac{r^2}{2} + A$$

where the constant A is determined from the condition that at $r = R$, we must have $v = 0$. This condition gives

$$A = \frac{R^2 \Delta p}{4\eta\ell}$$

Thus, we obtain the distribution of velocity in the tube as a function of the radius r as

$$v = \frac{R^2 \Delta p}{4\eta\ell} \left(1 - \frac{r^2}{R^2} \right) \tag{4.12}$$

We see that at the axis $r = 0$, the flow speed is $\frac{R^2 \Delta p}{4\eta \ell}$; let us denote this as v_0:

$$v_0 = \frac{R^2 \Delta p}{4\eta \ell} \tag{4.13}$$

Taking this into account, we finally get the flow speed distribution as

$$v = v_0 \left(1 - \frac{r^2}{R^2} \right) \tag{4.14}$$

This distribution is shown in Fig. 15.

To get the volume of liquid flowing out per unit time $\dot{Q}$, we find the fluid per second coming out from an area between radius r and $r + \mathrm{d}r$ along a cross section. This is $(2\pi r \mathrm{d}r)v$. Integrating this from $r = 0$ to $r = R$, we obtain the volume rate of flow

$$\dot{Q} = \int_{r=0}^{r=R} (2\pi r \mathrm{d}r)v = 2\pi v_0 \int_{r=0}^{r=R} \left(r - \frac{r^3}{R^2} \right) \mathrm{d}r = \frac{\pi R^2 v_0}{2}$$

Plugging in the expression for v_0 from Eq. (4.13), we finally obtain the volume rate of flow through the tube as

$$\dot{Q} = \frac{\pi R^4}{8\eta} \frac{\Delta p}{\ell} \tag{4.15}$$

This statement is known as *Poiseiulle's law* after the French physician and physiologist *Jean Louis-Marie Poiseiulle* (1799–1869) who discovered it because of his interest in blood flow.

4.3 A sphere falling through a fluid

A common situation where viscosity of the fluid plays an important role in the motion of a sphere in a fluid. A pendulum swinging in air, or a raindrop falling, or dust settling in water, are all examples.

As was derived by an English physicist *Sir George Gabriel Stokes*, it turns out that when a sphere of radius a falls in a fluid of viscosity η, the fluid exerts a resistive force given by (*Stoke's formula*):

$$\overrightarrow{\mathbf{F}} = -6\pi\eta a \overrightarrow{\mathbf{v}} \tag{4.16}$$

This is an interesting example of a retarding force proportional to the speed of the object. This retarding force is the reason why the oscillations of a pendulum becomes smaller and smaller and finally stops.

5 Surface Tension

At the interface between two materials, physical properties change rapidly over distances comparable to the molecular separation scale. Since a molecule at the interface is exposed to a different environment than inside the material, it will also have a different binding energy. In the continuum limit where the transition layer becomes a mathematical surface separating one material from the other, the difference in molecular binding energy manifests itself as a macroscopic *surface energy density*. And where energy is found, forces are not far away.

5.1 Basic physics of surface tension

The molecules in a fluid experience attractive and repulsive forces in all directions due to the surrounding molecules. Because the repulsive forces have a much shorter range than the attractive forces, and are important only at extremely high external pressures, attractive forces generally

predominate. The resultant force on such molecules in the bulk of the fluid, averaged over a macroscopic time (a time which is much longer than that spent in collisions between molecules), will be zero. The interaction of a given molecule with its (nearest) neighbors leads to a reduction of its potential energy, i.e., intermolecular forces act to stabilize the system. Such forces give rise to *cohesion* among the molecules of a liquid and to *adsorption* or *adhesion* between molecules of that liquid and any bounding solid surface.

The molecules at the surface region of the fluid have a smaller number of nearest neighbors (as the density of the gaseous region above the surface is considerably smaller than that of the bulk fluid), and therefore their potential energy is *not* decreased as much as in the interior of the fluid. Molecules near the surface of the fluid experience a weaker force from the gaseous region above the surface than they would if the gaseous region was replaced by fluid. Thus, such molecules will experience, on average, a force, *normal to the surface*, pulling them back into the bulk of the fluid. It is therefore necessary to do work against this force to take a molecule from the interior to the surface, and consequently the surface molecules possess greater energy than those inside the liquid. In other words, considering the potential energy of water molecules inside bulk water as zero, the molecules at the interface have a positive potential energy. This extra energy characterized as the *surface tension* (which we shall denote[1] by S), which is the interface potential energy divided by the interface area (J/m^2), which manifests itself as a force per unit length (N/m), the two units being the same.

5.1.1   Surface tension and force

To see the relation between the surface energy per unit area S and the force of the surface tension, consider a hypothetical curve C on the interface of liquid and air (see Fig. 16). If this curve is allowed to move orthogonally towards the right by an amount $\mathrm{d}s$, then the increment in the surface of the left part is $L\mathrm{d}s$ (where L is the length of the curve C). This leads to increase in
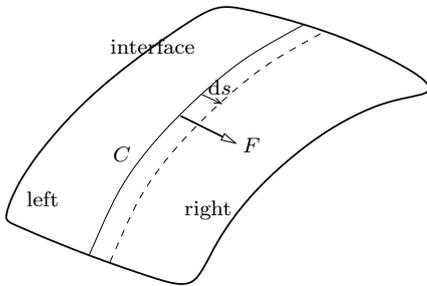
Figure 16: The force with which the piece of the surface on the right acts on the piece on the left across the curve element
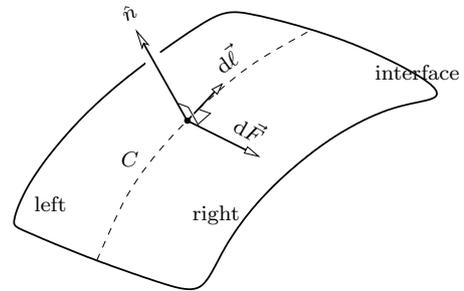
Figure 17: Defining the force of surface tension on the left part due to the right part.

the surface energy of the left part which is equal to $\mathrm{d}E = SL\mathrm{d}s$. This must be equal to the work done by the force of surface tension F exerted on the left part by the right part, so that

$$F\,\mathrm{d}s = SL\mathrm{d}s \quad \Rightarrow \boxed{F = SL}$$

To define surface tension force formally, let an oriented open surface be divided into two parts by an oriented curve, such that the surface has a uniquely defined left and right hand side with respect to the curve. If $\widehat{\mathbf{n}}$ denote the a unit vector normal to the surface, then

$$\boxed{\mathrm{d}\overrightarrow{\mathbf{F}} = S\,\mathrm{d}\overrightarrow{\boldsymbol{\ell}} \times \widehat{\mathbf{n}}} \tag{5.1}$$

---

[1]There is no universally agreed-upon symbol for surface tension which is variously denoted by σ, γ, α, S, Υ, and T.

is the force that the right hand side of the surface exerts on the left hand side through the curve element $\mathrm{d}\vec{\ell}$. The total force is obtained by integrating this expression along the curve.

At an interface between homogeneous materials, surface tension does not depend on how much the interface has already been stretched, and this makes the interface quite different from an elastic membrane which like the skin of a balloon resists stretching with increasing force because elastic tension increases as the deformation grows (Hooke's law). Soap films and biological membranes are an exception and behave like elastic membranes because of their peculiar surfactants. That is in fact what accounts for their great stability.

5.1.2 Why is surface tension force parallel to the surface

In a fluid in equilibrium the pressure can be defined as the average normal force per unit area exerted by all the molecules on one side of a small imaginary test surface in the liquid on all those on the other side. The total pressure can be separated into two parts:

(i) p_k, the kinetic contribution which arises due to the transport of momentum by molecules moving across the surface. This is the same way that the pressure in an ideal gas arises. In fact, for an ideal gas this is the only contribution to the pressure. The formula (from the ideal gas equation written down as)

$$p_k = N_u kT \tag{5.2}$$

(where N_u = number density of particles, k = Boltzmann's constant, and T = absolute temperature) is the same in a liquid, because the velocity distribution of molecules is the same for a liquid in equilibrium as it is for a gas. The kinetic pressure is, of course, essentially positive.

(ii) p_f, the static force pressure is due to the time average of the static forces between molecules on opposite sides of the test surface and is important only in a dense gas or a liquid. This static force pressure is essentially *negative* because under normal situations, the molecules attract each other. However, if the external applied pressure becomes very large so that the molecules are forced very near to each other, they start repelling each other and then this p_f becomes positive.

Thus, the total pressure

$$\underset{\text{(usually positive)}}{p} = \underset{\text{(always positive)}}{p_k} + \underset{\text{(usually negative)}}{p_f} \tag{5.3}$$

In the bulk liquid and gas the pressure has the same value, p_0, whatever the orientation of the test surface employed in its definition. Near the liquid surface, however, the tangential pressure p^t and the normal pressure p^n need not be the same, since there is no longer any symmetry of direction. But the fluid is in equilibrium, so that the forces on the opposite faces of a small cube must be equal and opposite (Fig. 18). This means that (apart from a cumulative downward increase due to gravity, which is negligible near the surface) the normal pressure must have the constant value p_0, right through the surface layer. The tangential pressure p^t, however, while being equal to p_0 in the bulk liquid and vapour, can and does vary in the surface layer.

To see this it is necessary first to understand how p^n can keep its constant value p_0, even though both its kinetic and static force contributions p_k^n and p_f^n vary with z, a coordinate perpendicular to the liquid surface which increases in value into the gas. From Eq. (5.2), the decrease in density with increasing z means that p_k^n decreases; the corresponding increase in p_f^n from negative values to nearly zero, necessary to keep p^n constant (Fig. 19), occurs because there are fewer pairs of molecules attracting across the test surface as it nears the liquid surface. Now p^t will have the same kinetic part p_k^t as p^n but its static force contribution p_f^t will increase towards zero
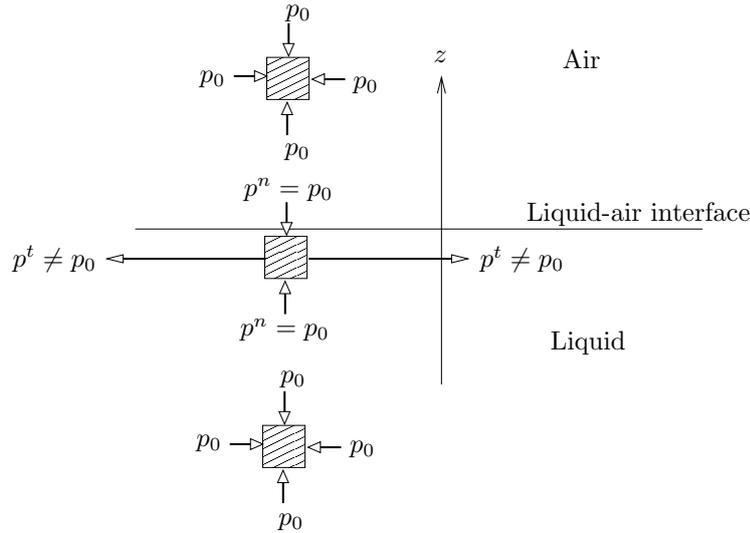
Figure 18: Equilibrium of elementary fluid cubes in liquid, vapour and surface.
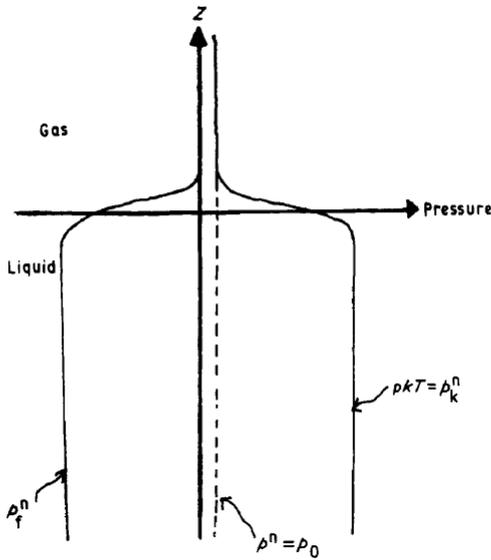

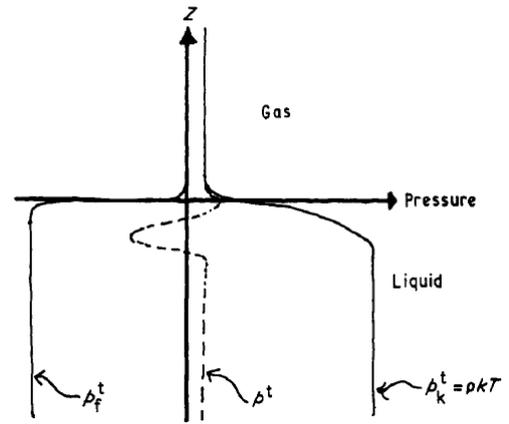Figure 19: Contributions to normal pressure.

Figure 20: Contributions to tangential pressure.

more slowly at first than p_f^n does; this is because the static force contribution comes mainly from molecules lying along perpendiculars through the test surface, so that the tangentially oriented test surfaces used to define p^n "see" the depletion of molecules in the liquid surface before the normally oriented test surfaces used to define p^t. If these two contributions to p^t are added (Eq. (5.3)), the resultant is not constant (Fig. 20), but goes negative so that there is, in fact, a *tangential tensile stress* near the liquid surface, which even for rare gas liquids, which cohere only weakly, can amount to more than a thousand atmospheres. Therefore, the surface layer of liquid, in contrast with the bulk, must possess rigidity in order to resist the shear stress that results from p^t differing from p^n; this is the reason why the liquid-air interface behaves like a stretched membrane.

The surface tension, S, defined in an elementary way as the total force exerted between the portions of liquid on opposite sides of a line of unit length in the surface, is simply the integral of the under-pressure through the surface layer, i.e.

$$S = \int_{-\infty}^{\infty} (p_0 - p^t(z)) \, \mathrm{d}z \tag{5.4}$$

5.2 Excess pressure

Everybody knows that the pressure inside a toy balloon is higher than outside, and that the excess pressure is connected with the tension in the taut balloon skin. Here we shall calculate the surface pressure discontinuity for a *drop* of a homogeneous liquid hovering weightlessly, for example in a spacecraft. In the absence of all external forces, surface tension will attempt to make the drop spherical because that shape has smallest area for a given volume.

Surface tension will also attempt to contract the drop but is stopped by the build-up of an extra positive pressure Δp, the pressure being higher in the concave region. To relate this excess pressure with the surface tension S, let's imagine that due to this excess pressure the boundary of the drop (whose radius is R) is pushed outwards by an amount $\delta R \ll R$. This leads to a surface energy increment of amount

$$\mathrm{d}E = 4\pi S((R+\delta R)^2 - R^2) = 4\pi S R^2 \left(\left(1 + \frac{\delta R}{R}\right)^2 - 1\right) \approx 4\pi S R^2 \left(1 + 2\frac{\delta R}{R} - 1\right) = 8\pi S R \delta R$$

This increment in energy comes at the expense of the work done by the force of the excess pressure. This work is

$$\delta W = \Delta p \, 4\pi R^2 \delta R$$

Work energy principle then implies $\mathrm{d}E = \delta W$, that is $8\pi S R \delta R = \Delta p \, 4\pi R^2 \delta R$. This gives the

$$\text{excess pressure for a liquid drop:} \quad \boxed{\Delta p = \frac{2S}{R}} \tag{5.5}$$

For a *bubble* (which is air trapped inside a thin envelope of liquid), since there are two liquid-air interfaces, the surface energy increment is double compared to that in the drop of liquid. For that matter, the excess pressure is double compared to that in the drop:

$$\text{excess pressure for a bubble:} \quad \boxed{\Delta p = \frac{4S}{R}} \tag{5.6}$$

♣ *Remark* 1. Eq. (5.5) is a special case of a more general result known as the *Young-Laplace* theorem which states that the excess pressure on the concave side of an interface

$$\Delta p = S\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{5.7}$$

where R_1 and R_2 are the *radii of curvatures* along *any two mutually perpendicular directions lying on the interface*. These will carry a plus or minus sign depending on whether the corresponding *centers of curvature* lie inside or outside that concave region.

5.3 Capiallary effects

5.3.1 Contact angle

When a liquid is brought into contact with a solid, *adhesion* of the liquid with the solid and *cohesion* of the liquid become competing forces. If the adhesion is more compared to the cohesion, the liquid tries to stick to the solid and is said to *wet* the solid. In this case, it will easily spread over the solid. On the other hand, if cohesion is more, the liquid does *not* wet the solid. These can also be stated in terms of the *contact angle* which is defined as the angle between the liquid surface and the solid, measured in the liquid, at the contact line – the location where the liquid surface meets the solid surface. If the contact angle is less that $90°$, the liquid is said to wet the surface, otherwise not (see Fig. 21 and 22). For example, water against a clean glass meet in a small acute contact angle $\theta \approx 0°$, whereas mercury meets glass at an obtuse contact angle
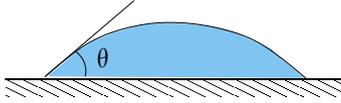
Figure 21: The liquid wets the solid for $\theta < 90°$.
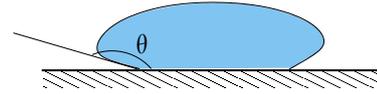
Figure 22: The liquid does not wet the solid for $\theta > 90°$.

of $\theta \approx 140°$. Due to its small contact angle, water is very efficient in wetting many surfaces, whereas mercury has a tendency to contract into pearls.

It should be emphasized that the contact angle is extremely sensitive to surface properties, fluid composition, and additives. This is especially true for the water-air-glass contact angle. In the household we regularly use surfactants that diminish both surface tension and contact angle, thereby enabling dishwater better to wet greasy surfaces on which it otherwise would tend to pearl. Oppositely, after washing our cars we apply a wax which makes rainwater pearl and prevents it from wetting the surface, thereby diminishing rust and corrosion.

5.3.2 Capillary rise

Water has a well-known ability to rise against the gravity above the ambient level in a narrow, vertical, cylindrical glass tube that is lowered into the liquid. This is called the *capillary effect.* It happens basically because adhesive forces are greater compared to the cohesive forces. On the contrary, mercury for example goes down the surrounding level in such a tube. For mostly
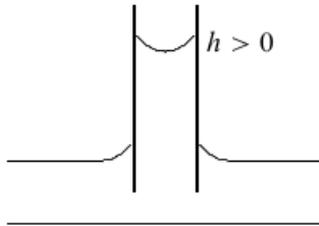

Figure 23: Water rises above the ambient level in a glass tube and displays a concave meniscus inside the tube.
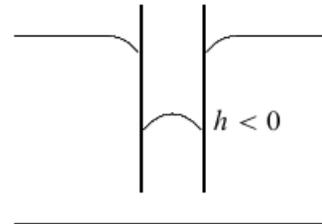

Figure 24: Mercury sinks below the general level in a capillary glass tube and displays a convex meniscus.

wetting liquids, the rise may also be seen as caused by the acute contact angle at the water-air-glass contact line which makes the surface inside the tube concave, such that the center of curvature lies outside the liquid. Surface tension will create a negative pressure just below the liquid surface, and that is what lifts the water column. Mercury with its obtuse contact angle displays instead a convex surface shape, creating a positive pressure jump which makes the liquid drop down to a level where the pressure just below the surface equals the ambient hydrostatic pressure at the same level outside. To calculate the capillary rise, consider the diagram shown in Fig. 25. The concave meniscus is part of a sphere with radius R that is related to the radius r of the capillary tube as $R\cos\theta = r$, where θ is the contact angle. Since the pressure at point 2 should be same as atmospheric pressure p_a, we get (neglecting the small height difference in the meniscus itself)

$$p_a = p_2 = p_1 + \rho h g$$

However, due to the fact that there is an excess pressure across a curved interface, we also have

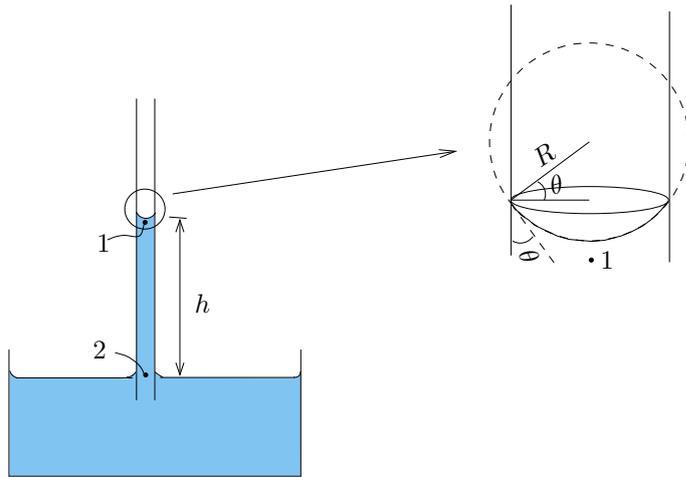$$p_a - p_1 = \frac{2S}{R} = \frac{2S\cos\theta}{r}$$

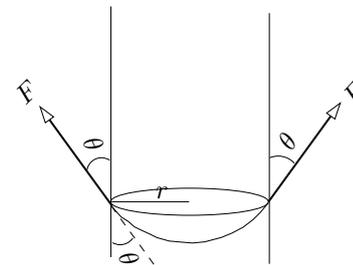Figure 25: Calculating the capillary rise using the pressure difference across the meniscus.

Figure 26: Calculating the capillary rise using the surface tension force.

From these two equations, we get

$$\rho g h = \frac{2S\cos\theta}{r} \quad \Rightarrow \boxed{h = \frac{2S\cos\theta}{\rho g r}} \tag{5.8}$$

Alternately, we can calculate the height using the fact that when equilibrium height is achieved, the weight of the liquid column will be supported by the vertical component of the surface tension force. Referring to Fig. 26, the surface tension force acts along the circular periphery diverging at an angle θ. So the net upward force is $2\pi r S\cos\theta$. This will balance the weight of the column which is $\rho \pi r^2 h g$ (neglecting the small amount of liquid below the curved meniscus. Hence,

$$2\pi r S\cos\theta = \rho \pi r^2 h g \quad \Rightarrow h = \frac{2S\cos\theta}{\rho g r}$$

which is the same result as has been obtained above.

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