Electromagnetism

1 Motional emf

The physical agencies which can provide a seat for emf are many: in a battery it's a chemical force; in a piezoelectric crystal mechanical pressure is converted into an electrical impulse; in a thermocouple it is the temperature difference that does the job; in a photoelectric cell it is light; and in a Van de Graaff generator the electrons are literally loaded onto a conveyor belt and swept along. But this list does not include the most common seat of emf: a *generator*. Generators exploit **motional emf**'s, which arises when *you move a wire through a magnetic field.*

Figure 1 shows a primitive model for a generator. In the shaded region there is a uniform magnetic field $\vec{B}$, pointing into the page, and the resistor R represents whatever is (maybe a light bulb or a toaster) through which we are trying to drive the current. If the entire loop is pulled

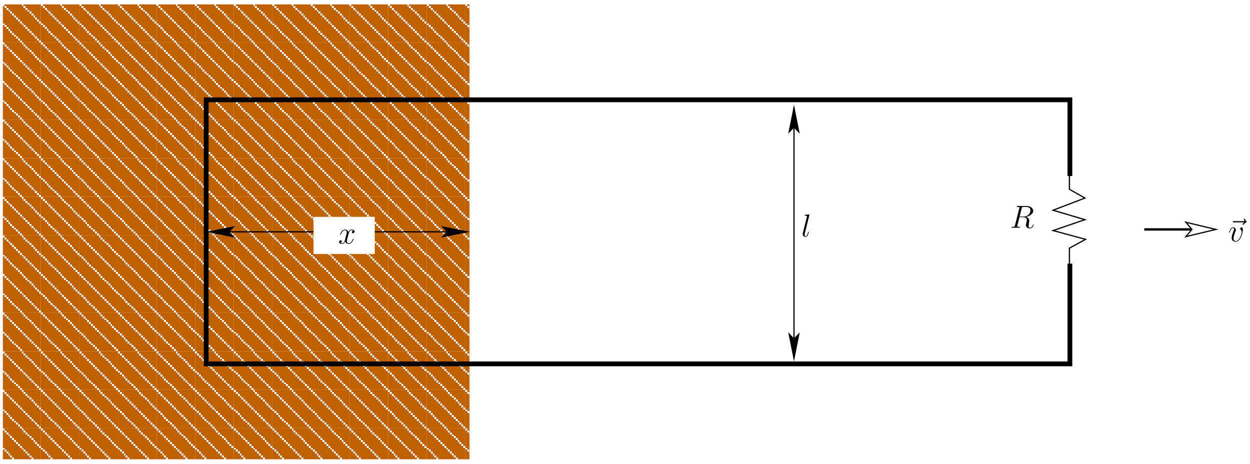


Figure 1: A loop of wire moving through a magnetic field.

to the right with speed v, the charges in segment ab experience a magnetic force whose vertical component qvB drives current around the loop, in the clockwise direction. The emf, being the work done per unit charge becomes:

$$\mathcal{E} = \frac{1}{q}\oint \vec{F}_{mag} \cdot d\vec{l} = vBh, \tag{1.1}$$

where $\vec{F}_{mag}$ is the magnetic force and h is the width of the loop. (The horizontal segments bc and ad contribute nothing since the force here is perpendicular to the wire.)

Notice that the integral you perform to calculate $\mathcal{E}$ (Equation 1.1) is carried at *one instant of time* — take a "snapshot" of the loop, if you like, and work from that. Thus $d\vec{l}$, for the segment

ab in Figure 1, points straight up, even though the loop is moving to the right. You can't quarrel with this — it's simply the way emf is *defined* — but it *is* important to be *clear* about it.

In particular, although the magnetic force is responsible for establishing the emf, it is certainly not doing any work — magnetic forces *never* do any work. Who, then, *is* supplying the energy that heats the resistor? *Answer*: The person who is pulling the loop! With the current flowing, charges in segment ab have a vertical velocity (call it $\vec{u}$) in addition to the horizontal velocity $\vec{v}$ they inherit from the motion of the loop. Accordingly, the magnetic force has a component quB to the left. To counteract this, the person pulling on the wire must exert a force on each charge equal to

$$F_{pull} = quB$$

to the *right* (see Figure 2). This force is transmitted to the charge by the structure of the wire.

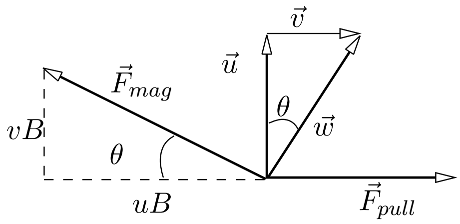


Figure 2

Meanwhile, the particle is *moving* in the direction of the resultant velocity $\vec{w}$, and the distance it goes is $(h/\cos\theta)$. The work done per unit charge is therefore

$$\frac{1}{q}\int \vec{F}_{pull}\cdot d\vec{l} = uB\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$

($\sin\theta$ coming from the dot product). As it turns out, then, the *work done per unit charge is exactly equal to the emf*, though the integrals are taken along entirely different paths and completely different forces are involved. To calculate the emf you integrate around the loop at *one instant*, but to calculate the work done you follow a charge in its motion around the loop; $\vec{F}_{pull}$ contributes to the emf, because it is perpendicular to the wire, whereas $\vec{F}_{mag}$ contributes nothing to the work because it is perpendicular to the motion of the charge.

There is a particularly nice way of expressing the emf generated in a moving loop. Let Φ_B be the flux of $\vec{B}$ through the loop:

$$\Phi_B \equiv \int \vec{B}\cdot d\vec{A}. \tag{1.2}$$

For the rectangular loop in Figure 1,

$$\Phi_B = Bhx.$$

As the loop moves, the flux decreases:

$$\frac{d\Phi_B}{dt} = Bh\frac{dx}{dt} = -Bhv.$$

(The minus sign accounts for the fact that dx/dt is negative.) But this is precisely the emf; evidentally the emf generated in the loop is minus the rate of change of flux through the loop:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}. \tag{1.3}$$

This is the **flux rule** for motional emf. Apart from its delightful simplicity, it has the virtue of applying to *non*rectangular loops moving in *arbitrary* directions through *non*uniform magnetic fields; in fact, the loop need not maintain a fixed shape.

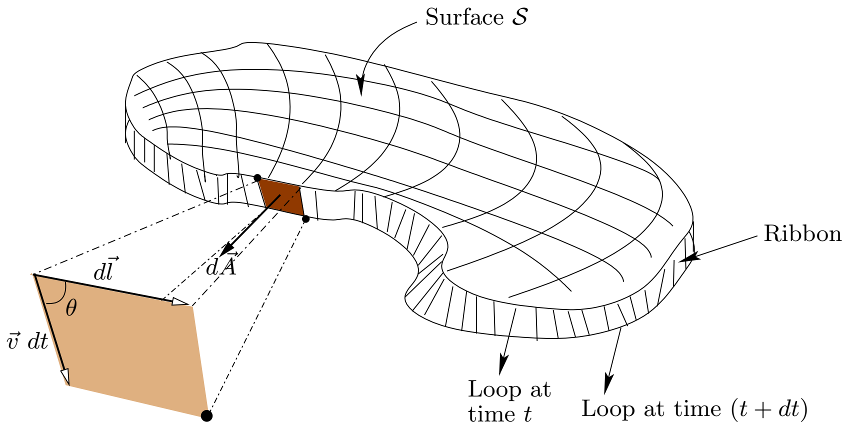


Figure 3: A loop moving in a magnetic field.

Proof. Figure 3 shows a loop of wire at time t and also a short time dt later. Suppose we compute the flux at time t, using the surface $\mathcal{S}$, and the flux at time $t+dt$, using the surface $\mathcal{S}$ plus the "ribbon" that connects the new position of the loop to the old one. The *change* in flux, then, is

$$d\Phi_B = \Phi_B(t+dt) - \Phi_B(t) = \Phi_B^{ribbon} = \int_{ribbon} \vec{B}\cdot d\vec{A}.$$

Focus your attention on point P: in time dt it moves to P'. Let $\vec{v}$ be the velocity of the *wire*, and $\vec{u}$ the velocity of a point charge *down* the wire;$\vec{w}=\vec{v}+\vec{u}$ is the resultant velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$d\vec{A} = (\vec{v}\times d\vec{l})\ dt.$$

Therefore

$$\frac{d\Phi_B}{dt} = \oint \vec{B}\cdot(\vec{v}\times d\vec{l}).$$

Since $\vec{w}=\vec{v}+\vec{u}$ and $\vec{u}$ is parallel to $d\vec{l}$, we may write the last expression as

$$\frac{d\Phi_B}{dt} = \oint \vec{B}\cdot(\vec{w}\times d\vec{l}).$$

Now, the scalar triple–product can be written :

$$\vec{B}\cdot(\vec{w}\times d\vec{l}) = -(\vec{w}\times\vec{B})\cdot d\vec{l},$$

so that

$$\frac{d\Phi_B}{dt} = -\oint (\vec{w}\times\vec{B})\cdot d\vec{l}.$$

But $(\vec{w}\times\vec{B})$ is the magnetic force per unit charge, $\vec{F}_{mag}/q$, so

$$\frac{d\Phi_B}{dt} = -\frac{1}{q}\oint \vec{F}_{mag}\cdot d\vec{l},$$

and the integral on the right is nothing but the emf. Hence,

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

□

The calculation of emf round a circuit requires that we compute the flux. While calculating the flux, which direction does the area vector $d\vec{A}$ should point? This ambiguity in the direction of area vector is resolved (as always) by the right hand rule: If your fingers (of the right hand) curl along the positive direction around the loop, then your thumb indicates the direction of $d\vec{A}$. Should the emf come out negative, it means the current flows in the opposite direction around the loop.

The flux rule is a nifty shortcut for calculating motional emf's. It does not contain any new physics. Occasionally, you might run across problems that cannot be handled by the flux rule; for these one must go back to the Lorentz force law itself.

***Example* 1.** A metal disc of radius R rotates with angular velocity ω about a vertical axis, through a uniform magnetic field $\vec{B}$ with $\vec{B} \uparrow\uparrow \vec{\omega}$. Find the emf developed between the disc's center and its perimeter.

Solution:The speed of a point on the disc at a distance r from its center is $v = r\omega$, so that the force per unit charge is $|\vec{v} \times \vec{B}| = \omega r B$ and it is directed radially away. The emf is therefore

$$\mathcal{E} = \int_0^a \omega r B\, dr = \omega B \int_0^a r\, dr = \frac{1}{2}\omega B a^2.$$

The trouble with the flux rule is that it assumes the current flows along a well–defined path, where as in this example (if you connect a resistance across the axle and the perimeter) the current spreads out over the whole disc. It's not even clear what the "flux through the circuit" would *mean* in this context. Even more tricky is the case of **eddy currents**. Take a chunk of aluminum (say), and shake it around a non–uniform magnetic field. Current will be generated in the material, and you will feel a kind of "viscous drag" — as though you were pulling the aluminum sheet through a thick jelly. This effect is because of eddy currents which circulate in the aluminum sheet. Eddy currents are notoriously difficult to calculate, but easy and dramatic to demonstrate.

2 Electromagnetic Induction

2.1 Faraday's Laws

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1 He pulled a loop of wire to the right through a magnetic field (Figure 4(a)). A current flowed in the loop.

Experiment 2 He moved the *magnet* to the *left*, holding the loop still (Figure 4(b)). Again, a current flowed in the loop.

Experiment 3 With both the loop and the magnet at rest, he changed the *strength* of the field with time (using a electromagnet, varying the current in the coil, Figure 4(c)). Once again, current flowed in the loop.

The first experiment, of course, is an example of motional emf, conveniently expressed by the flux rule. I don't think it will surprise you to learn that exactly the same emf arises in experiment 2 — all that really matters is the *relative* motion of the magnetic field and the loop. In experiment 3, nothing is moving around but still a current is induced. How come? Is something *changing*. Yes, the magnetic *flux* (because of the changing magnetic field) through the loop. But even a changing magnetic field cannot exert a force on *stationary* charges. The only other force which a charge — whether stationary or moving — can feel is the *electric* force. Thus in the third case, the *current must have been caused by an electric force* and that requires an electric field to be present. Faraday has an ingenious inspiration:

A changing magnetic field* induces *an electric field.

It is this "induced" field that is responsible for the emf in Experiment 2.[1] Indeed, if (as Faraday

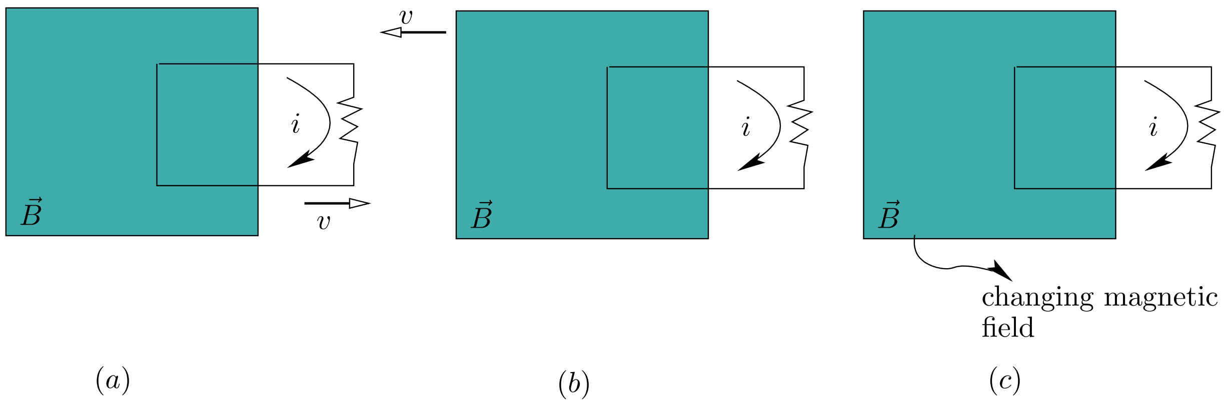


Figure 4: Faraday's Experiments.

found empirically) the emf is again equal to the rate change of the flux,

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}, \tag{2.1}$$

then $\vec{E}$ is related to the change in $\vec{B}$ by the equation

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \tag{2.2}$$

This is **Faraday's law**.

In Experiment 3 the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi_B/dt$. Indeed one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{2.3}$$

will appear in the loop.

Many people call *this* the Faraday's law. But actually there are really *two different* mechanisms underlying Equation 2.3, and to identify both of them as the "Faraday's law" is a little like saying that because identical twins look alike, we ought to call them by the same name. In Faraday's first experiment, it is the Lorentz force law at work; the emf is *magnetic*. But in the other two, it is an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the *Special theory of Relativity* — he sought a deeper understanding of what seemed like a peculiar phenomena in classical electrodynamics. In the meantime, we shall use the term "Faraday's law" for electric fields induced by changing magnetic fields, and as such Experiment 1 is *not* an instance of Faraday's law.

***Example* 2.** A long solenoid is moving through a conducting loop of slightly larger diameter with a constant velocity v as shown in Figure 5. The solenoid carries a current i moving in a clockwise direction as seen in the direction of its velocity and its length is L which is very large compared to its radius a and the number of turns per unit length is n. Graph the emf induced in the ring as a function of time.

[1] You might argue that the magnetic field in Experiment is not really *changing* — just *moving*. What I mean is that if you sit at a *fixed location*, the field *does* change, as the magnet passes by.

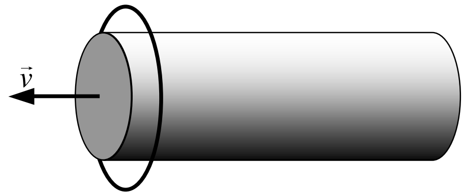
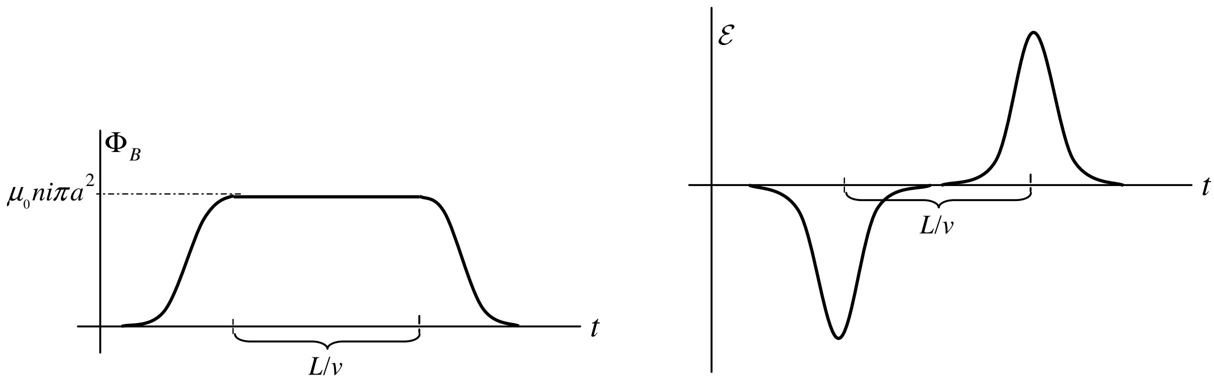


Figure 5: Example 2 (the dimension of the ring is highly exaggerated)

Solution: The magnetic field of the solenoid points along the same direction as the velocity vector and is more or less uniform inside it but near the end it spreads out. The flux through the ring is zero when the solenoid is far away; it it builds up to a maximum of $\mu_0 ni\pi a^2$ as the leading end passes the loop; and it drops back to zero as the trailing end emerges. The emf is (minus) of the derivative of Φ_B with respect to time, so it consists of two spikes as shown in Figure 6(b).



(a) The variation of the flux through the loop in Example 2

(b) The variation of induced emf through the loop in Example 2

Figure 6: Solution: Example 2

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in the above example we would like to know which *way* around the loop, the induced current flows. In principle, the right-hand thumb rule does the job (we call Φ_B positive to the left, in Figure 6(a), so that the positive direction for the current in the ring is counterclockwise, as viewed from the left; since the first spike is Figure 6(b) is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise) But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:

The induced current in a loop will flow in such a direction that the flux **it** *produces, tends to cancel the change in the flux through that loop.*

If the circuit is not closed, the Lenz's law still gives the direction of the induced emf *supposing that the current was to flow.* Notice that it is not the flux which the induced current is opposing but the *change* in flux. Faraday's induction is a kind of "inertial" phenomenon: A conducting loop "likes" to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (Though it doesn't *succeeds* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

***Example* 3. The "jumping ring".** If you wind a solenoid coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on the top, and switch the current on, the ring will jump several feet in air (see Figure 7). Why?

Solution: *Before* you turned on the current, the flux through the loop was *zero*. *Afterward* a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz's law, was in such a direction that *its* field tends to cancel this new flux. This means that the current in the loop is *opposite* to the current in the solenoid. And opposite currents repel, so the ring flies off.

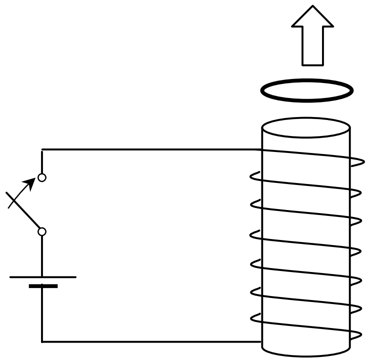


Figure 7: The "jumping ring" demonstration.

2.1.1 The Induced Electric Field

What Faraday's discovery tells us is that there are really two distinct kinds of electric field: those attributable directly to electric charges, and those associated with changing magnetic fields. The former can be calculated (in the static case) using Coulomb's law; the latter can be found by exploiting the analogy between Faraday's law,

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A},$$

and Ampère's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{\text{enclosed}}.$$

Of course(!), the line integral (or the *circulation*) is not enough to determine a vector field — you must also specify the surface integral (the *flux*). But as long as $\vec{E}$ is a *pure* Faraday field, due exclusively due to a changing $\vec{B}$ (meaning that there are no charges around to produce their electr*static* field), Gauss's law says

$$\oint \vec{E}\cdot d\vec{A} = 0$$

for any closed surface, since there are no charges, while for magnetic field, of course

$$\oint \vec{B}\cdot d\vec{A} = 0$$

always. So the parallel is complete, and we can conclude that *Faraday's induced electric fields are determined by* $-(\partial\vec{B}/\partial t)$ *in exactly the same way as magnetostatic fields are determined by* $\mu_0 i_{\text{enclosed}}$.

In particular, if symmetry permits, we can use all tricks associated with Ampère's law, only this time we use Faraday's law.

***Example* 4.** A uniform magnetic field $\vec{B}(t)$, pointing straight up, fills the shaded circular region of Figure 8. If $\vec{B}$ is changing with time, what is the induced electric field?

Solution: $\vec{E}$ points in the circumferential direction, just like the *magnetic* field inside a long straight wire carrying a uniform *current* density. Draw an Amperian loop of radius r, and apply Faraday's law:

$$\oint \vec{E}\cdot d\vec{l} = E(2\pi r) = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\pi r^2 B(t)\right) = -\pi r^2 \frac{dB}{dt}.$$

Therefore,

$$\vec{E} = -\frac{r}{2}\frac{dB}{dt}\hat{\phi}.$$

If $\vec{B}$ is *increasing*, $\vec{E}$ runs *clockwise*, as viewed from above.

***Example* 5.** A line charge of linear density λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in Figure 9, so that it is free to rotate (the spokes are made of some nonconducting material — say wood). In the central region, out to radius a, there is a uniform magnetic field $\vec{B}_0$, pointing up. Now someone turns the field off. What happens?
Solution: The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the the charge at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that the *resulting current's* magnetic field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Quantitatively, Faraday's law says

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\pi a^2 \frac{dB}{dt}.$$

Now, the torque on a segment of length $d\vec{l}$ is $(\vec{r}\times\vec{F})$, or $b\lambda E dl$. The total torque on the wheel is therefore

$$\tau = b\lambda \oint E\ dl = -b\lambda\pi a^2 \frac{dB}{dt},$$

and the angular momentum imparted to the wheel is

$$\int \tau dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0.$$

It doesn't matter how fast or slow you turn off the field; the ultimate angular velocity of the wheel is the same regardless. (Wonder from where the angular momentum *came* from?!)

A final word on this example: It's the *electric* field that did the rotating. To convince you of this, the entire things were deliberately set so that the *magnetic* field is always *zero* at the location of the charge (the rim).

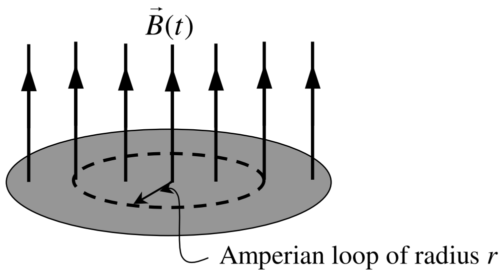


Figure 8: Example 4

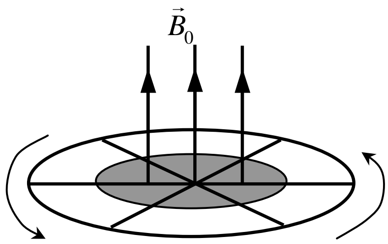


Figure 9: Example 5

I must warn you, now, of a small fraud that takes away some beauty from many applications of Faraday's law: Electromagnetic induction, of course, occurs only when the magnetic fields are *changing*, and yet we would like to use the tools of magneto*statics* (Ampère's law, the Biot–Savart law, and the rest) to *calculate* those magnetic fields. Technically, any result derived in this way is only approximately correct (this regime is appropriately called **quasistatic**). But in practice the error is usually negligible unless the field fluctuates extremely rapidly, or you are interested in points very far from the source. Apart from these considerations, we can always use the tools already mentioned (so don't let this humble warning bother you!). Generally speaking, it is only when we come to electromagnetic waves and radiations that we must worry seriously about the breakdown of magnetostatics itself.

2.1.2 Inductance

Suppose you have two loops of wire, at rest (Figure 10). If you run a steady current i_1 around loop 1, it produces a magnetic field $\vec{B}_1$. Some of the field lines pass through loop 2; let Φ_2 be the flux of $\vec{B}_1$ through 2. You might have a tough time actually *calculating* $\vec{B}_1$, but a glance at the

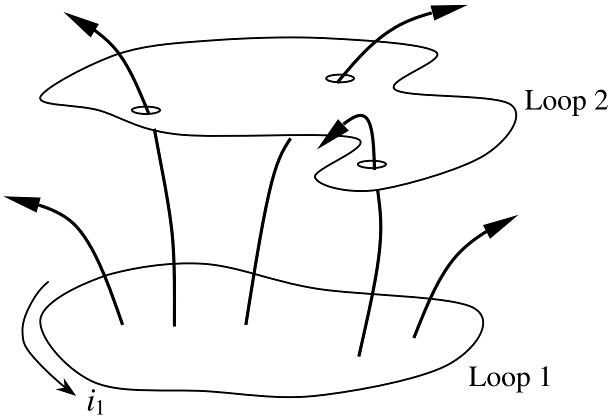


Figure 10: The current through loop 1 creates a flux through loop 2

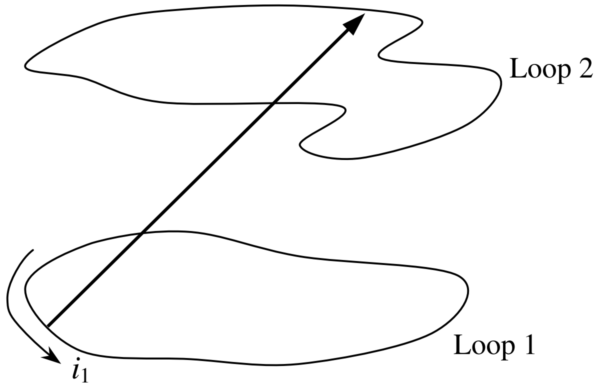


Figure 11: Using the Neumann Formula

Biot–Savart law,

$$\vec{B}_1 = \frac{\mu_0}{4\pi} i_1 \oint \frac{d\vec{l}_1 \times \hat{r}}{r^2},$$

reveals one significant fact about this field: *It is proportional to the current* i_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \vec{B}_1 \cdot dA_2.$$

Thus

$$\Phi_2 = M_{21} i_1, \tag{2.4}$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a formula, which I will state without any derivation. It is called the **Neumann formula** and gives the mutual inductance of two loops (see Figure 11):

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}. \tag{2.5}$$

As is obvious from its nature, it is not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral in Equation 2.5 is unchanged if we switch the role of loops 1 and 2; it follows that

$$M_{21} = M_{12}. \tag{2.6}$$

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through* 2 *when we run a current* i *around* 1 *is identical to the flux through* 1 *when we send the same current* i *around* 2. We may as well drop the subscripts and call them both M.

***Example* 6.** A short solenoid (length l and radius a with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Figure 12. Current i flows in the short solenoid. What is the flux through the long solenoid? *Solution*: Since the inner solenoid is short, it

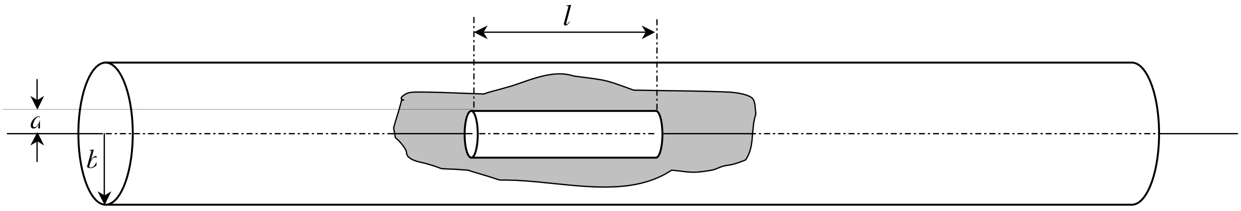


Figure 12: Example 6

has a very complicated field; moreover, it puts a different amount of flux through each turn of the outer solenoid. It would be very *miserable* to compute the total flux through this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current i through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 i,$$

so the flux through a single loop of the short solenoid is:

$$B\pi a^2 = \mu_0 n_2 i \pi a^2.$$

There are $n_1 l$ turns in all, so the total through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 i l.$$

This is also the flux, a current i in the *shorter* solenoid would put through the *longer* one, which is what we set out to find. Incidentally, the mutual inductance, in this case, is

$$M = \mu_0 \pi a^2 n_1 n_2 l.$$

Suppose now that you *vary* the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says that this changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{di_1}{dt}. \qquad (2.7)$$

What a remarkable thing: Every time you change the current in loop 1, an induced current flows in loop 2 — even though there are no wires connecting the two.

Now consider this: a changing current in loop 1 not only induces an emf in loop 2, but it also induces an emf in loop 1 *itself* since the flux through loop 1 is also changing in this process (and it does not matter that the flux is changing because of *whose* field). Once again, the field and therefore the flux is proportional to the current:

$$\Phi = Li. \qquad (2.8)$$

The constant of proportionality L is called the **self–inductance** (or simply the **inductance**) of this loop. As with M, it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in this loop is:

$$\mathcal{E} = -L\frac{di}{dt}. \qquad (2.9)$$

Inductance (*and* mutual inductance) are measured in **henries** (symbol: H); a henry is a volt–second per ampere.

***Example* 7.** Find the self inductance of a toroid with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns. (Figure 13)
Solution: Let a current i be sent through the windings of the toroid. The magnetic field inside the toroid is

$$B = \frac{\mu_0 N i}{2\pi r}.$$

The flux through a single turn (Figure 13) is

$$\int \vec{B} \cdot d\vec{A} = \frac{\mu_0 N i}{2\pi} h \int_a^b \frac{dr}{r} = \frac{\mu_0 N i h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The *total* flux is N times this, so the inductance becomes

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right).$$

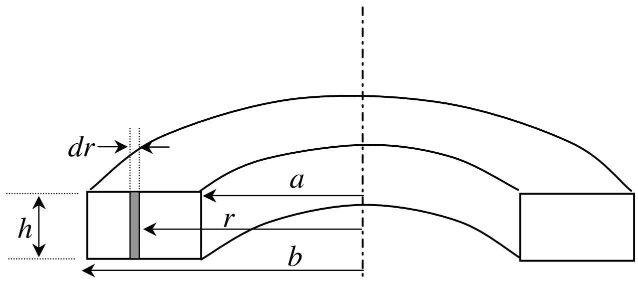


Figure 13: Example 7

Inductance (like capacitance) is an intrinsically *positive* quantity. Lenz's law, which is enforced by the minus sign in Equation 2.9, dictates that the emf is in such a direction as to *oppose* the *change in the current*. For this reason, it is called a **back emf**. Whenever you try to alter the current in a wire, you must fight against this back emf. Thus inductance plays somewhat the same role in electric circuits that *mass* does in the the mechanical systems: The greater the L is, the harder it is to change the current, just as the larger the mass is, the harder it is to change its velocity.

***Example* 8.** Suppose a steady current I is flowing around a loop when somebody suddenly cuts the wire. The current drops "instantaneously" to zero. This generated a whopping back emf, for although the current itself may be small, the rate at which it falls is enormous and as such the back emf is also enormous. That's why you often draw a spark when you unplug a toaster or iron — electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.

Nothing so dramatic happens when you plug *in* the toaster. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and continuous buildup. Suppose, for instance, that a battery (which supplies a constant emf $\mathcal{E}_0$) is connected to a circuit consisting of resistance R and inductance L (see Figure 14) by closing the switch at $t = 0$. Find the current as a function of time.
Solution: The total emf in this circuit is that provided by the battery plus the back emf provided by the inductance. Ohm's law says

$$\mathcal{E}_0 - L\frac{di}{dt} = iR,$$

which can also be written as

$$\frac{di}{dt} + \frac{R}{L} i = \frac{\mathcal{E}_0}{L}.$$

You can verify that the solution in this case is given by

$$i(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right).$$

This function is plotted in Figure 15. Had there been no inductance in the circuit, the current in the circuit would have jumped immediately to the value of $\mathcal{E}_0/R$ which is the steady state value. In practice, *every* circuit has *some* self inductance. and the current approaches its steady state value asymptotically. The quantity $\tau \equiv L/R$ is called the (inductive) **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly two–third) of its steady state value.

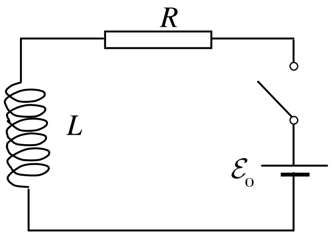


Figure 14: An LR circuit.

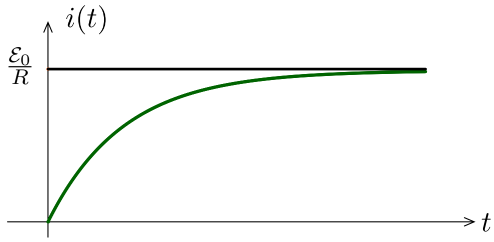


Figure 15: Variation of current through the LR circuit.

2.1.3 Energy in Magnetic Field

It takes a certain amount of *energy* to start a current flowing in a circuit. I'm not talking about the energy delivered to the resistors and converted into heat–that is irretrievably lost as for as the circuit is concerned and can be large or small, depending on how long you let the current run. What I am concerned with, rather, is the work you must do *against the back emf* to get the current going. This is *fixed* amount, and it is *recoverable*: you get it back when the current is turned off. In the mean time it represents the energy latent in the circuit.

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done *by you against* the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is i. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}i = Li\frac{di}{dt}.$$

If we start with zero current and build it up to a final value of i, the work done is (integrating the last equation over time):

$$W = \frac{1}{2}Li^2. \tag{2.10}$$

It does not depends on how *long* we take to build the current, only on the geometry of the loop (in the form of L) and the final current i.

We can think of this energy as being stored in the magnetic field of the inductor in the same way that the energy of the capacitor was stored in the electric field. Supposing that the formula for a infinitely long solenoid holds, we have

$$B = \mu_0 ni$$

and

$$L = \mu_0 n^2(Al),$$

following usual notations. Eliminating L, n and i from these equations, we get the energy as

$$U = W = \frac{1}{2\mu_0}B^2(Al).$$

Now Al is the volume of the solenoid, so the magnetic energy density u_B (that is magnetic energy per unit of volume) becomes,

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}. \tag{2.11}$$

Though derived under a special circumstance, the above formula is valid in general. In view of this result, we say that the any point where a magnetic field exists can be thought of a site of stored magnetic energy with a density given by Equation 2.11. At the same time, the concept of energy density also offers an alternate way to find out the inductance: find the energy using the formula for energy density and equate it to $(1/2)Li^2$.

You might find it strange that it takes energy to set up a magnetic field — after all the magnetic fields *themselves* do no work. The point is that producing a magnetic field, where there was none earlier, requires *changing* the magnetic field which induces an *electric* field. The latter, of course, *can* do work.