Time Varying Voltages & Currents

1 Electric Oscillations

Suppose you charge a capacitor (capacitance C) and then connect its plates to each other through an inductor coil (inductance L). What will happen? The process has been depicted in Figure 1.

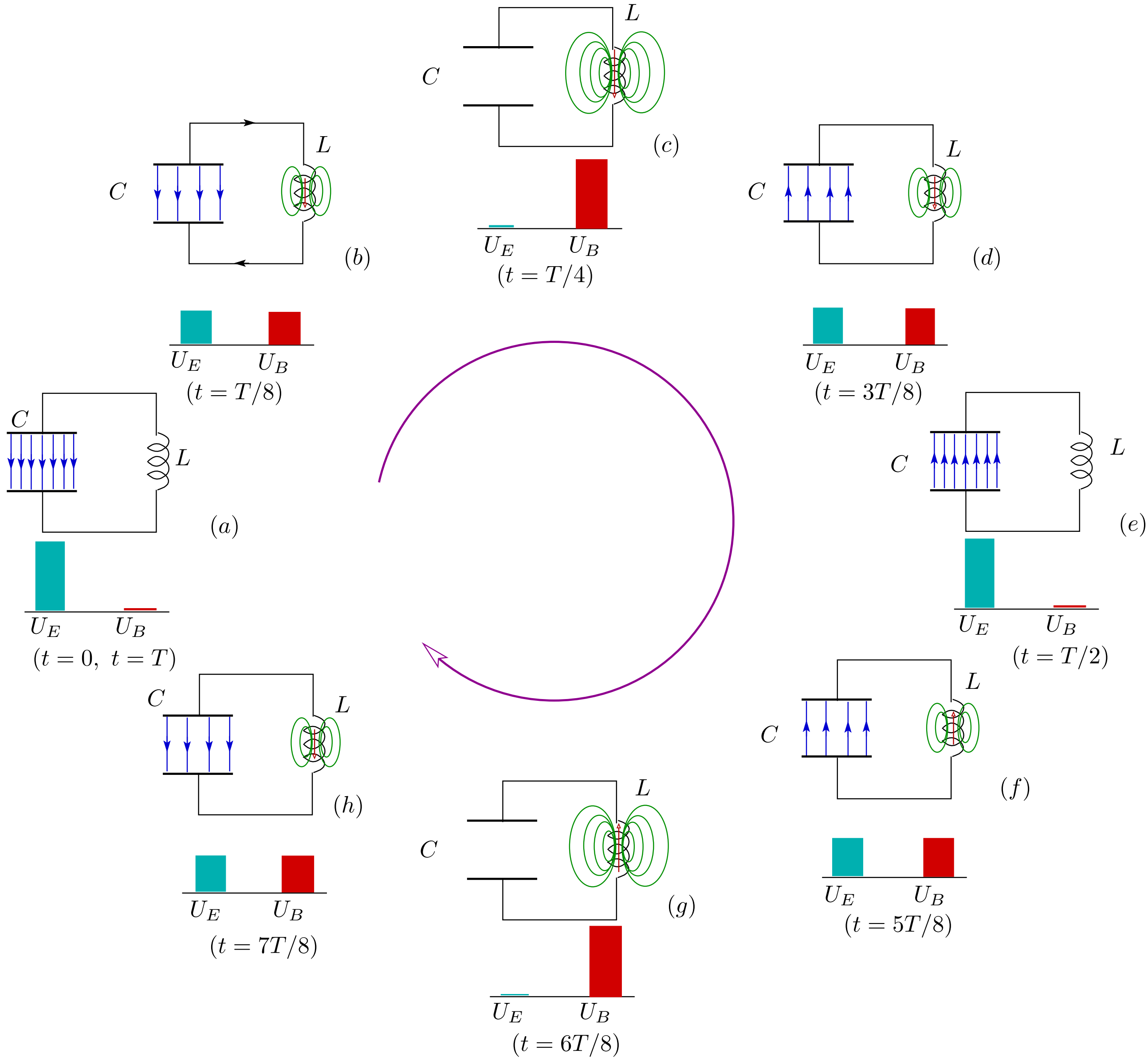

Figure 1: The *free* oscillations in an LC circuit for one complete cycle. The bar graphs show the distribution of electric and magnetic energy at different instants of time. Note that by the time one complete oscillation of the charge (*or* current) has taken place, the energy in either C or L has undergone two oscillations.

Suppose that initially, the upper plate of the capacitor is charged positively and the lower plate negatively (Figure 1(a)). In this case, the entire energy of the circuit is concentrated in the capacitor. As soon as the capacitor has been connected to the inductor, it starts to discharge, and a current flows through the coil L. Owing to the opposition by the back emf of the inductor, the current rises slowly. The electrical energy of the capacitor is converted to the magnetic energy of the coil. This process terminates when the capacitor is discharged completely, while the current in the circuit attains the maximum value (Figure 1(c)) At this instant the entire electrical energy has been converted into the magnetic energy of the inductor. Starting from this moment, the current starts to decrease retaining its direction. However, it does not ceases immediately since it is sustained by the self–induced emf in the inductor. The current recharges the capacitor but this time the polarity of the capacitor is opposite that of the initial situation.At the same time, the appearing electric field tends to reduce the current. Finally, the current ceases, while the capacitor is fully charged (Figure 1(e)). The entire energy is once more electrical. From this moment, the capacitor starts to discharge again, the current flows in the opposite direction, and the process is repeated (Figure 1(f) through (a)) until the state of the circuit is restored to the initial state. The entire cycle begins again only to be repeated again and so on.

If the conductors constituting the oscillatory circuit have no resistance, strictly periodic *oscillations will be observed in the circuit.* In the course of the process, the charge on the capacitor plates, the voltage across the capacitor and the current in the inductor coil vary periodically. The oscillations are accompanied by mutual conversion of the energy of electric and magnetic fields.

However, *if the resistance of the circuit* $R \neq 0$, then, in addition to the process described, *electromagnetic energy will be transformed into Joule's heat.* The circuit in this case exhibits *damped oscillations.*

1.1 Equation of an Oscillatory Circuit

Let us consider a circuit containing series connected capacitor C, inductor coil L, resistor R and a time varying external voltage source $V(t)$ (Figure 2). Let us first choose the positive direction

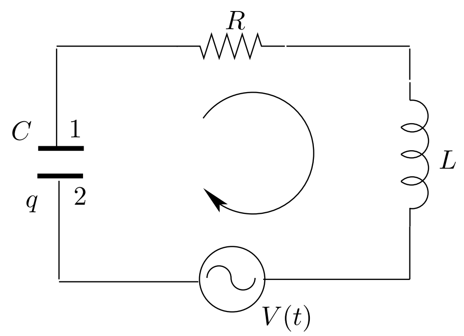

Figure 2: An LCR circuit

for traversing the loop as clockwise (you could take it as anti–clockwise as well). We denote by q the charge on the capacitor plate the direction from which to the other plate coincides with the chosen direction of our circumvention of the loop. Then the current in the circuit is defined as

$$i = \frac{dq}{dt} \tag{1}$$

Consequently, if $i > 0$, then $dq > 0$ as well, and vice versa.

In accordance with Ohm's law for section $1RL2$ of the circuit, we have

$$iR = V_1 - V_2 + \mathcal{E}_s + V(t), \tag{2}$$

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where $\mathcal{E}_s$ is the self induced emf in the inductor coil. In the case under consideration,

$$\mathcal{E}_s = -L\frac{di}{dt} \qquad \text{and} \qquad V_2 - V_1 = \frac{q}{C}.$$

Hence Equation 2 can be written in the form

$$L\frac{di}{dt} + iR + \frac{q}{C} = V(t), \tag{3}$$

or taking into account (1),

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}\,q = V(t). \tag{4}$$

This is the *equation of an oscillatory circuit*, which is a linear second order nonhomogeneous differential equation with constant coefficients. Using this equation for calculating $q(t)$, we can easily obtain the voltage across the capacitor as $V_c = V_2 - V_1 = q/C$ and current i by formula (1).

The above equation can be given a different form:

$$\frac{d^2q}{dt^2} + 2\gamma\frac{dq}{dt} + \omega_0^2 q = \frac{1}{L}V(t), \tag{5}$$

where the following notation is introduced:

$$2\gamma = \frac{R}{L}, \qquad \omega_0^2 = \frac{1}{LC}. \tag{6}$$

The quantity ω_0 is called the *natural frequency* of the circuit and γ is the *damping factor*. The meaning of these quantities will be explained in due course.

If $V(t) = 0$, i.e. the circuit contains no source, the oscillations are called *free oscillations*. They will be *undamped* for $R = 0$ and *damped* for $R \neq 0$. We consider all these cases.

1.2 Free Oscillations

1.2.1 Free Undamped Oscillations

If a circuit contains no external voltage and if its resistance $R = 0$, the oscillations in such a circuit will be *free* and *undamped*. The equation describing these oscillations is a particular case of Equation 5 when $V(t) = 0$ and $R = 0$:

$$\frac{d^2q}{dt^2} + \omega_0^2\, q = 0. \tag{7}$$

The solution of this equation is the function

$$q = q_0\cos(\omega_0 t + \alpha), \tag{8}$$

where q_0 is the maximum value of the charge on capacitor plates, ω_0 is the natural frequency of the oscillatory circuit, and α is the initial phase. The value of ω_0 is determined only by the properties of the circuit itself, while the values of $q_0$and α depend on the initial conditions. For these conditions we can take, for example, the value of the charge q and current $i = \frac{dq}{dt}$ at the moment $t = 0$.

According to Equation 6, $\omega_0 = 1/\sqrt{LC}$; hence the period of free undamped oscillations is given by

$$T_0 = 2\pi\sqrt{LC}. \tag{9}$$

Having found current i by differentiating Equation 8 with respect to time and bearing in mind that the voltage across the capacitor plates is in phase with charge q, we can easily see that in free undamped oscillations current i leads, in phase, the voltage across the capacitor plates by $\pi/2$.

While solving certain problems, energy approach can also be used.

1.2.2 Free Damped Oscillations

Every real oscillatory circuit has a resistance, and the energy stored in the circuit is gradually spent on heating. Free oscillations will be damped. We can obtain the equation for a given oscillatory circuit by putting $V(t) = 0$ in Equation 5. This gives

$$\frac{d^2q}{dt^2} + 2\gamma\frac{dq}{dt} + \omega_0^2 q = 0. \tag{10}$$

I shall give an idea as how to solve this homogeneous differential equation. Form the *auxiliary equation* by replacing the differential operator d/dt by x and treating the second derivative as exponentiation:

$$x^2 + 2\gamma x + \omega_0^2 = 0 \tag{11}$$

The solution of this equations are called the *eigenvalues* of the equation. Once we know the eigenvalues of the homogeneous equation, the *most general solution* is given as:

$$q = Ae^{x_1 t} + Be^{x_2 t}, \tag{12}$$

where x_1, x_2 are the eigenvalues; A and B are constants. To get any particular solution, we will need any two pieces of information to eliminate the constants A and B. This information is generally provided in terms of the initial conditions.

Solving (11) for our case, we have the solutions as

$$x = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

from where we get the eigenvalues as

$$x_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}, \qquad x_2 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}. \tag{13}$$

The following cases arises:

1. $\gamma > \omega_0$: In this case the values of x_i's are real and thus from (12) we see that both terms are decaying exponentials. Hence in this case, the charge q just exponentially falls to zero. In this case, the circuit is not oscillatory: it's *over-damped* and the discharge of the capacitor is *aperiodic.*

2. $\gamma = \omega_0$: It is the case of *critical damping.* We have a repeated solution which is once again not oscillatory.

3. $\gamma < \omega_0$: This represents *under-damped* situation. In this case the roots are complex and the solution turns out to be oscillatory. We study this solution.

For case 3 above, we write the solutions as:

$$x_{1,2} = -\gamma \pm \imath\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm \imath\omega_d,$$

where $\imath = \sqrt{-1}$ and ω_d is the *damped frequency*:

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}. \tag{14}$$

After little manipulations, the solution in this case takes the form

$$q = q_0 e^{-\gamma t}\cos(\omega_d t + \alpha), \tag{15}$$

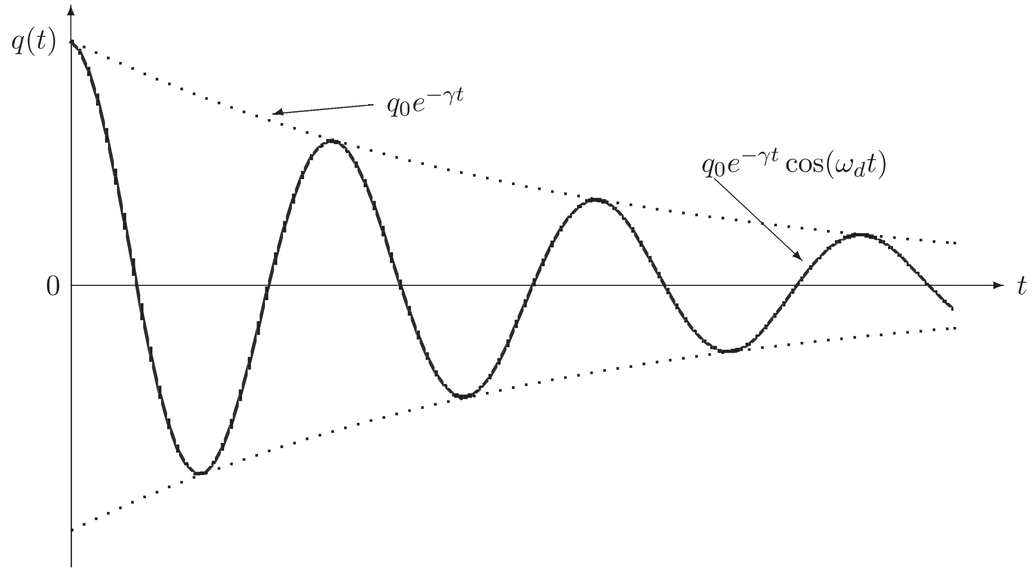

Figure 3: The variation of the charge q as a function of time for a damped and free oscillations. The graph shown is of the function (15) with α set to zero and it shows a cosine wave sandwiched between a decaying exponential.

where q_0 and α are arbitrary constants determined from the initial conditions. The plot of the function (15) is shown in Figure 3. It can be seen that this is not a periodic function since it determines damped oscillations. Nevertheless, the quantity $T = 2\pi/\omega_d$ is called the *period of damped oscillations*:

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} = \frac{T_0}{\sqrt{1-(\gamma/\omega_0)^2}}, \tag{16}$$

where T_0 is the period of free undamped oscillations.

The factor $A(t) = q_0e^{-\gamma t}$ in (15) is called the *amplitude of damped oscillations*. It's dependence on time is shown in Figure 3 by the dashed line.

Voltage across a Capacitor and Current in an Oscillatory Circuit. Knowing $q(t)$, we can find the voltage across a capacitor and the current in a circuit. The voltage across a capacitor is given by

$$V_c = \frac{q}{C} = \frac{q_0}{C}e^{-\gamma t}\cos(\omega_d t + \alpha). \tag{17}$$

The current in the circuit is

$$i = \frac{dq}{dt} = q_0e^{\gamma t}[-\gamma\cos(\omega_d t + \alpha) - \omega_d\sin(\omega_d t + \alpha)].$$

We transform the expression in the brackets to cosine. For this purpose, we multiply and divide this expression by $\sqrt{\omega_d^2 + \gamma^2} = \omega_0$ and then introduce angle δ by the formulas

$$-\frac{\gamma}{\omega_0} = \cos\delta, \qquad \frac{\omega_d}{\omega_0} = \sin\delta. \tag{18}$$

After this, the expression for i becomes:

$$i = \omega_d q_0 e^{-\gamma t}\cos(\omega_d t + \alpha + \delta). \tag{19}$$

It follows from (18) that angle δ lies in the second quadrant ($\pi/2 < \delta < \pi$). This means that in the case of a non–zero resistance, the current in the circuit *leads* (in phase) the voltage across the capacitor by more than $\pi/2$. It should be noted that for $R = 0$, this lead is $\delta = \pi/2$.

1.2.3 Quantities Characterizing Damping

The following quantities characterize any damped circuit.

1. *Damping coefficient* γ and *relaxation time* τ, that is the time during which the amplitude of oscillations decreases by a factor of e. It can be easily seen from (15) that

$$\tau = \frac{1}{\gamma}. \tag{20}$$

2. *Logarithmic decrement* λ *of damping.* It is defined as the Natural log of two successive values of the amplitudes measured in a period T of oscillations:

$$\lambda = \ln \frac{A(t)}{A(t+T)} = \gamma T. \tag{21}$$

3. *Quality factor* Q of an oscillatory circuit is, by definition,

$$Q = \frac{\pi}{\lambda}. \tag{22}$$

where λ is the logarithmic decrement. For the LCR circuit, we can write the quality factor as:

$$Q = \frac{\pi}{\gamma T} = \frac{\omega_d}{2\gamma} = \frac{L}{R}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}. \tag{23}$$

For a weak damping $\gamma^2 \ll \omega_0^2$ which means $\omega_d \approx \omega_0$ the quality factor becomes

$$Q \approx \omega_0 \frac{L}{R}. \tag{24}$$

One more useful form of quality factor for *weak damping* is:

$$Q \approx 2\pi \frac{U}{\Delta U}, \tag{25}$$

where U is the energy stored in the circuit and ΔU is the decrease in this energy during the period T of oscillations.

2 Forced Electric Oscillations. Alternating Current

Steady State Oscillations. We return to the Equations 3 and 5 for an oscillatory circuit and consider the case when the circuit includes an external voltage which vary with time harmonically:

$$V(t) = V_m \cos \omega t, \tag{26}$$

where V_m is the amplitude and ω is the frequency of the voltage. This voltage law occupies a special place owing to the properties of the oscillatory circuit itself to retain a harmonic form of oscillations under the action of external harmonic voltage. Further, the standard power production is of this kind of voltage and current. Before, we take up the solution of the oscillatory circuit in this case, we first define some terms and introduce some standard notations.

2.1 Definitions

1. ***Average*** or ***Mean Value*:** For any continuous function $f(x)$, we define the average value f_{av} over an interval $[a,\ b]$ as:

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\ dx. \tag{27}$$

Geometrically, f_{av} is the height of the rectangle constructed on the base ab such that its area is equal to the area (algebraic) under the graph of the function $f(x)$ bounded by the x axis, the straight line $x = a$ and $x = b$ (Figure 4). Accordingly, for a harmonically varying

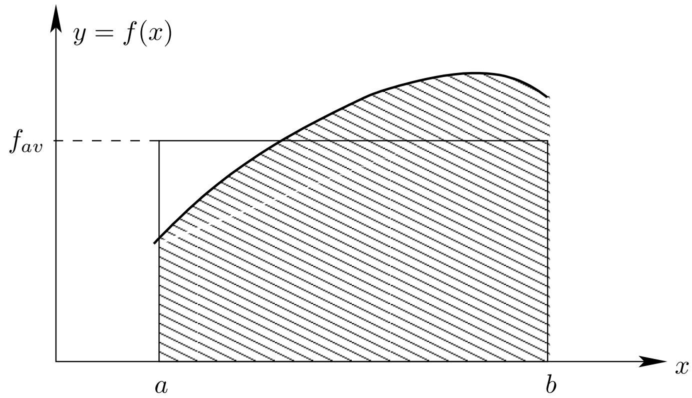

Figure 4: Geometrical Interpretation of Average value of a function.

voltage or current,

$$V_{av} = 0.$$

2. ***Root Mean Square Value*:** As the name suggests, the root mean square or RMS is the *square root of the mean of the squares* of the quantities. Accordingly,

$$f_{rms} = \sqrt{\frac{1}{b-a}\int_a^b f^2(x)\ dx}. \tag{28}$$

In particular, for harmonic function $V(t) = V_m \cos\omega t$ for one complete period T,

$$\begin{aligned} V_{rms} &= \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2\omega t\ d} = \sqrt{\frac{1}{T}\cdot\frac{V_m^2}{2}\int_0^T 2\cos^2\omega t\ dt} \\ &= \sqrt{\frac{1}{T}\cdot\frac{V_m^2}{2}\int_0^T (1+\cos 2\omega t)\ dt} \\ &= \sqrt{\frac{1}{T}\cdot\frac{V_m^2}{2}\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T} \\ &= \sqrt{\frac{1}{T}\cdot\frac{V_m^2}{2}\cdot T} = \frac{V_m}{\sqrt{2}} \end{aligned}$$

Hence,

$$\text{for a harmoic voltage,}\qquad V_{rms} = \frac{V_m}{\sqrt{2}}. \tag{29}$$

3. ***Complex Notation.*** **The $\jmath$ operator:** From Euler's formula, $e^{\imath\theta} = \cos\theta + \imath\sin\theta$. Using standard electrical engineering notation, let us denote by $\jmath = \sqrt{-1}$. Then from Euler's formula, we have $e^{\jmath\theta} = \cos\theta + \jmath\sin\theta$. Utilizing this formula, we define a ***complex*** voltage:

$$\tilde{\mathbf{V}} = V_m \cos\omega t + \jmath V_m \sin\omega t = V_m e^{\jmath\omega t}, \tag{30}$$

keeping in mind that the *real voltage* is the *real part* of $\tilde{\mathbf{V}}$:

$$V(t) = \mathrm{Re}\left(\tilde{\mathbf{V}}\right). \tag{31}$$

In this complex notation, the voltage $\tilde{\mathbf{V}}$ can be represented as a *phasor* (a vector)in the Complex plane which rotates with a *uniform angular speed* of ω. The length/modulus of this phasor is the amplitude V_m of the voltage and its instantaneous phase is ωt.

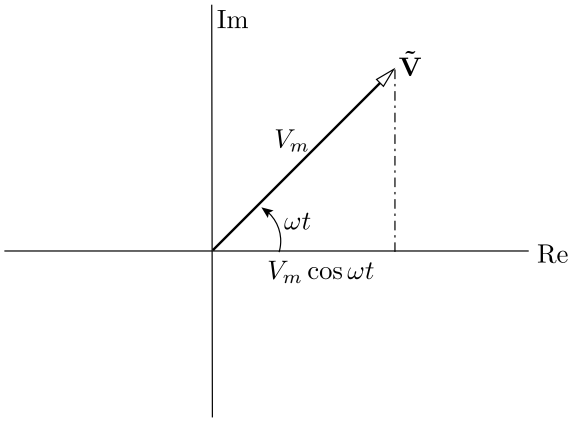

Figure 5: Phasor diagram for harmonic voltage.

4. ***Reactance and Impedance.*** In an ac circuit, the capacitors and inductors also pose resistance to the current. The *reactance* of the capacitor and the inductor is defined as:

$$\text{Capacitive Reactance: } X_C = \frac{1}{\omega C}, \qquad \text{Inductive Reactance: } X_L = \omega L. \tag{32}$$

Correspondingly, we define the impedances as:

$$\text{Capacitive Impedance: } \tilde{\mathbf{Z}}_{\mathbf{C}} = \frac{1}{\jmath\omega C} = -\frac{\jmath}{\omega C}, \qquad \text{Inductive Impedance: } \tilde{\mathbf{Z}}_{\mathbf{L}} = \jmath\omega L. \tag{33}$$

The benefit of defining these impedances is that while handling the complex quantities, we add impedances like resistances and the Ohm's law ca be used in the same form as was used for real current; only now the complex quantities replace the real ones.

In polar form, $\tilde{\mathbf{Z}}_{\mathbf{C}} = X_C e^{-\jmath\frac{\pi}{2}}$ and $\tilde{\mathbf{Z}}_{\mathbf{L}} = X_C e^{\jmath\frac{\pi}{2}}$. The nice thing about this complex approach is that impedances can be combined just like resistors.

2.2 Solving an Oscillatory Circuit for Harmonic Input

Figure 6 shows the LCR circuit, where the complex quantities have been indicated. The source is a harmonic voltage $V(t) = V_m \cos\omega t$. We use the method of phasor to solve it. The total

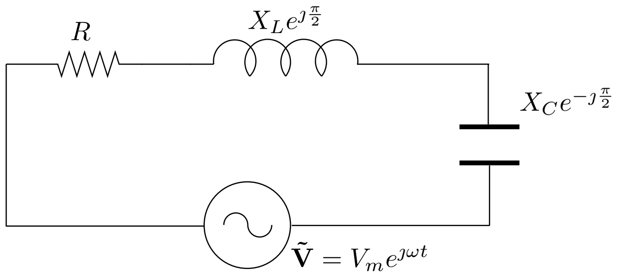

Figure 6: An LCR Circuit

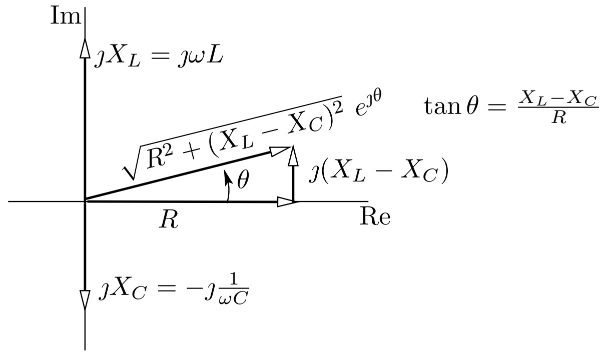

Figure 7: Phasor representation of the impedance in the complex plane.

impedance of the circuit is:

$$\begin{aligned}\tilde{\mathbf{Z}} &= R + \jmath\omega L - \jmath\frac{1}{\omega C} \\ &= R + \jmath\left(\omega L - \frac{1}{\omega C}\right) = R + \jmath(X_L - X_C) \\ \text{or} \qquad \tilde{\mathbf{Z}} &= \sqrt{R^2 + (X_L - X_C)^2}\ e^{\jmath\theta} \end{aligned} \tag{34}$$

where the phase angle θ is given by

$$\tan\theta = \frac{X_L - X_C}{R}. \tag{35}$$

Applying Ohm's law, we get the complex current $\tilde{\mathbf{I}}$ as:

$$\tilde{\mathbf{I}} = \frac{\tilde{\mathbf{V}}}{\tilde{\mathbf{Z}}} = \frac{V_m e^{\jmath\omega t}}{\sqrt{R^2 + (X_L - X_C)^2}\ e^{\jmath\theta}}$$

Thus, we get

$$\tilde{\mathbf{I}} = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}\ e^{\jmath(\omega t - \theta)} = I_m\ e^{\jmath(\omega t - \theta)}, \tag{36}$$

where we have put

$$I_m = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}, \tag{37}$$

which is the *current amplitude*. The real current $i(t)$ in the circuit is just the real part of $\tilde{\mathbf{I}}$, namely:

$$i(t) = I_m \cos(\omega t - \theta). \tag{38}$$

We study different possible cases for the above circuit.

1. $L = 0, C = \infty$ ($X_C = 0$), $R \neq 0$ (**Purely resistive circuit**): In this case, the impedance of the circuit is just R. Consequently, the phase angle $\theta = 0$ and the current amplitude $I_m = V_m/R$. The real current is $i(t) = (V_m/R)\cos\omega t$. Thus we conclude that in a purely resistive circuit,

the current and voltage are in phase.

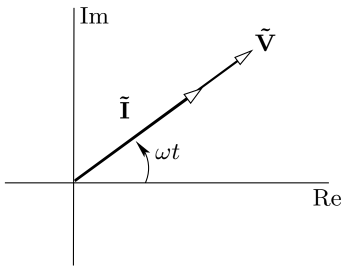

Figure 8: Voltage and current phasor for a purely resistive circuit.

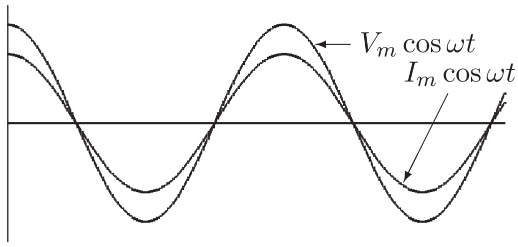

Figure 9: The variation of voltage and current in a purely resistive circuit. Ohm's law is valid at all instants of time.

The voltage and the current phasor are shown in Figure 8 while the variation of voltage and current as a function of time is shown in Figure 9. Note that in this case, the Ohm's law is valid at all instants of time[1].

2. $L = R = 0$ and the capacitance has some finite value (**Purely capacitive**): In this case, the complex impedance is $\tilde{\mathbf{Z}} = X_C e^{-\jmath\pi/2}$ whose modulus is just X_C. As such the phase angle $\theta = -\pi/2$ and the real current becomes $i(t) = \omega C V_m \cos(\omega t + \pi/2)$ with the result that

   the current <u>leads</u> the voltage (in phase) by 90° in a purely capacitive circuit.

   The voltage and the current phasor are shown in Figure 10 while the variation of voltage and current as a function of time is shown in Figure 11. Note that the current amplitude $I_m = \omega C V_m$ increases as we increase the frequency.

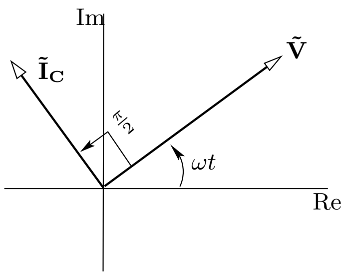

Figure 10: Voltage and current phasor for a purely capacitive circuit.

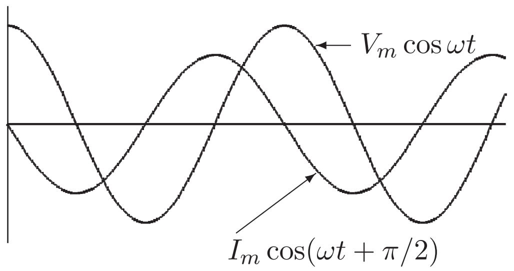

Figure 11: The variation of voltage and current in a purely capacitive circuit.

3. $R = 0,\ C = \infty\ (X_C = 0),\ L \neq 0$ (**Purely Inductive Circuit**): For this case, the complex impedance is $\tilde{\mathbf{Z}} = X_L e^{\jmath\pi/2}$ whose modulus is just X_L. As such the phase angle $\theta = \pi/2$ and the real current becomes $i(t) = (V_m/\omega L)\cos(\omega t - \pi/2)$. Conclusion:

   the current <u>lags</u> behind the voltage (in phase) by 90° in a purely inductive circuit.

[1]This *assumption* holds true only at low frequency. At high frequency, the resistance *does* depends slightly on the frequency.

The voltage and the current phasor are shown in Figure 12 while the variation of voltage and current as a function of time is shown in Figure 13.

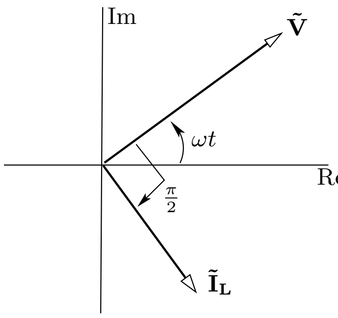

Figure 12: Voltage and current phasor for a purely inductive circuit.

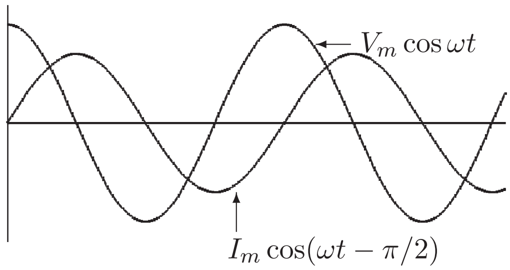

Figure 13: The variation of voltage and current in a purely inductive circuit.

2.3 Power Considerations

The power supplied by the source is given by

$$P(t) = V(t)i(t) = (V_m \cos \omega t)(I_m \cos(\omega t - \theta)) = V_m I_m \cos \omega t \cos(\omega t - \theta). \tag{39}$$

Using trigonometric formula

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B),$$

we get from (39),

$$P(t) = \frac{1}{2} V_m I_m [\cos(2\omega t - \theta) + \cos \theta] = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} [\cos(2\omega t - \theta) + \cos \theta],$$

which is the same as

$$P(t) = V_{rms} I_{rms} [\cos(2\omega t - \theta) + \cos \theta]. \tag{40}$$

Thus the power varies as a function of time. The important consideration in an AC circuit is, however, the average power supplied (or absorbed). In the above equation, the first term in the brackets is a cosine whose average over one complete cycle is zero. Thus the average power supplied by the source is:

$$P_{av} = V_{rms} I_{rms} \cos \theta. \tag{41}$$

In words: ***The power supplied to the circuit is the product of the rms value of voltage and the rms value of the*** **in phase component** ***of the current.***

The term $\cos \theta$ is called the ***power factor*** of the circuit. In terms of the circuit elements, the power factor can be written as

$$\cos \theta = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{42}$$

We note that for a *purely resistive* circuit, $\cos \theta = 1$ and thus the power absorbed by the circuit is $V_{rms} I_{rms}$, while for a *purely inductive* as well as a *purely capacitive* circuit, $\cos \theta = 0$ and thus *a purely inductive/capacitive circuit does not absorb any power*! Can you explain this?

2.4 Back to LCR circuit

Going to the standard solution for the LCR circuit (Figure 6). The solution for the circuit, as we have already found out, for a sinusoidal voltage $\tilde{\mathbf{V}} = V_m\ e^{\jmath\omega t}$ is given by:

$$\tilde{\mathbf{I}} = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}\ e^{\jmath(\omega t - \theta)} = I_m\ e^{\jmath(\omega t-\theta)},$$

where

$$\tan\theta = \frac{X_L - X_C}{R}.$$

The complex voltage across the resistor will be, by Ohm's law:

$$\tilde{\mathbf{V}}_{\mathbf{R}} = \tilde{\mathbf{I}}R.$$

Similarly, the complex voltage across the inductor and capacitor are respectively:

$$\tilde{\mathbf{V}}_{\mathbf{L}} = \tilde{\mathbf{I}}\tilde{\mathbf{Z}}_{\mathbf{L}}, \qquad \text{and} \qquad \tilde{\mathbf{V}}_{\mathbf{C}} = \tilde{\mathbf{I}}\tilde{\mathbf{Z}}_{\mathbf{C}}.$$

Adding all these complex voltages,

$$\begin{aligned} \tilde{\mathbf{V}}_{\mathbf{R}} + \tilde{\mathbf{V}}_{\mathbf{L}} + \tilde{\mathbf{V}}_{\mathbf{C}} &= \tilde{\mathbf{I}}\left(R + \tilde{\mathbf{Z}}_{\mathbf{C}} + \tilde{\mathbf{Z}}_{\mathbf{L}}\right) \\ &= \frac{\tilde{\mathbf{V}}}{R + \tilde{\mathbf{Z}}_{\mathbf{C}} + \tilde{\mathbf{Z}}_{\mathbf{L}}}\left(R + \tilde{\mathbf{Z}}_{\mathbf{C}} + \tilde{\mathbf{Z}}_{\mathbf{L}}\right) \\ &= \tilde{\mathbf{V}} \end{aligned}$$

Thus the *sum of the complex voltage drops across the individual elements is the source voltage*

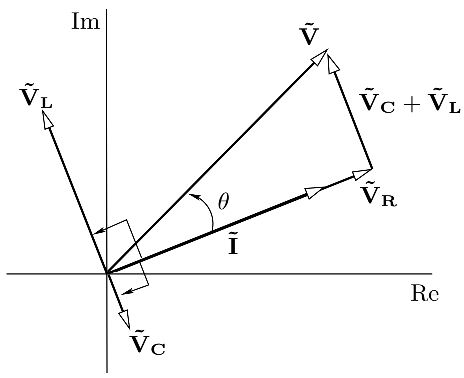

Figure 14: Voltage and current phasor for an LCR circuit assuming $X_L > X_C$.

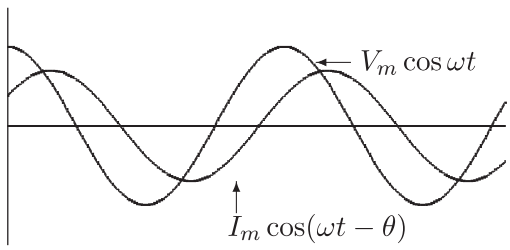

Figure 15: The variation of voltage and current in an LCR circuit assuming $X_L > X_C$.

as it must be. This is shown in Figure 14, assuming that $X_L > X_C$. Note that in this case, $\theta > 0$ is positive and hence the current lags behind the voltage (if θ had been negative the current will lead the voltage). For the same case, the variation of the real voltage and the current as a function of time is shown in Figure 15.

3 Resonance

Since both X_L and X_C are functions of the frequency ω of the source, the current amplitude I_m itself is a function of the frequency ω:

$$I_m(\omega) = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \tag{43}$$

From the above expression, we note that as the source frequency is changed, the current amplitude for the same circuit changes. Specifically, when the frequency is such that $X_L = X_C$ the impedance of the circuit is minimum (R) and as such the current amplitude maximum. At the same time, the phase angle $\theta = 0$ and the current is in phase with the voltage and *the circuit absorbed maximum power from the source*. This phenomena is called **resonance**. From the condition that $X_L = X_C$, we note that it takes place when the frequency of the source is exactly equal to the natural frequency ω_0 of the circuit. As such the natural frequency is also the *resonant* frequency:

$$\omega_r = \omega_0 = \frac{1}{\sqrt{LC}}. \tag{44}$$

The resonance curve for the current amplitude is shown in Figure 16. The maximum current amplitude at resonance is higher and the sharper, the smaller the damping factor $\gamma = R/2L$.

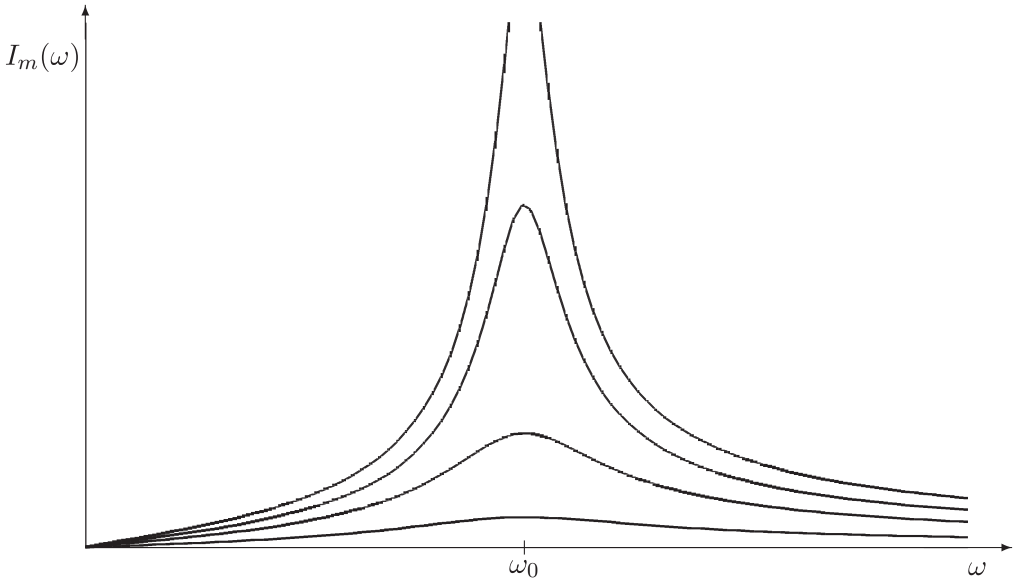

Figure 16: The variation of the current amplitude as a function of the frequency ω. The different graphs correspond to different values of the resistance R.

Resonance Curves and Quality factor. The shape of the resonance curve is connected in a certain way with the quality factor of the oscillatory circuit. This connection has the simplest form for the case of weak damping, i.e. for $\gamma^2 \ll \omega_0^2$. In this case, it turns out that

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{45}$$

where ω_0 is the resonant frequency and $\Delta\omega$ is the width of the resonance curve at a "height" equal to $1/\sqrt{2}$ of the peak height, i.e. at resonance.

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