Appendix P–11–A

Vectors

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1 Vectors

If you walk 4 km due north and then 3 km due east, you will have walked a total of 7 km, but you are *not* 7 km from where you set out, you are only 5 (verify it!). We need an arithmetic to describe quantities like this, which evidently do *not* add in the ordinary way. The reason they don't, of course, is that **displacements** (straight line segments going from one point to another) have *direction* as well as *magnitude* (length), and it is essential to take both into account when you add them. Such quantities are called **vectors**: velocity, acceleration, force, and momentum are other examples. By contrast, quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density, and temperature.

I shall denote vectors by small **boldface** letters ornamented with an arrowhead like $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$, $\overrightarrow{\mathbf{c}}$ and so on. The magnitude of a vector $\overrightarrow{\mathbf{a}}$ is written $|\overrightarrow{\mathbf{a}}|$ or simply a. In writing, you can write a vector simply by putting the arrowhead like $\overrightarrow{a}$, $\overrightarrow{b}$, and so on.

In diagrams, vectors are represented as arrows[1]: *the length of the arrow is proportional to the magnitude of the vector and the arrowhead represents the direction* (see Figure 1). Note that vectors have magnitude and direction but not *location*: a displacement of 4 km due north from New Delhi is represented by the same vector as a displacement 4 km due north in Durgapur (neglecting, of course, the curvature of the earth). On a diagram, therefore, you can slide the arrow around at will, as long as you don't change its length or direction.

1.1 Vector operations

As with any other mathematical objects, vectors require a set of rules so that we can manipulate them correctly. We define four vector operations: addition and three kinds of multiplication.

[1]Note that the arrow is not the vector itself; only a representation.

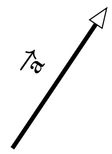


Figure 1: The arrow representation of a vector.

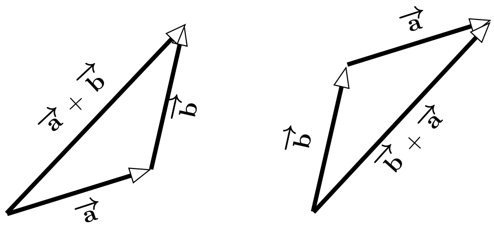


Figure 2: Defining the sum of two vectors. The vector sum is geometrical in nature. As can be seen, vector sum is *commutative.*

1.1.1 Addition of two vectors

To add two vectors $\vec{a}$ and $\vec{b}$ place the tail of $\vec{b}$ at the head of $\vec{a}$; the sum $\vec{a}+\vec{b}$, is the vector from tail of $\vec{a}$ to the head of $\vec{b}$ (Figure 2). This procedure can be generalized to find the sum of more than two vectors. To find the sum of vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4, \ldots,$ $\vec{a}_{n-1}$, and $\vec{a}_n$, place the tail of $\vec{a}_2$ at the head of $\vec{a}_1$, the tail of $\vec{a}_3$ at the head of $\vec{a}_2$, the tail of $\vec{a}_4$ at the head of $\vec{a}_3$ and so on and finally the tail of $\vec{a}_n$ at the head of $\vec{a}_{n-1}$. The sum

$$\vec{s} = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \ldots + \vec{a}_n$$

is, then, the vector from the tail of $\vec{a}_1$ to the head of $\vec{a}_n$ (Figure 3). The vector addition is

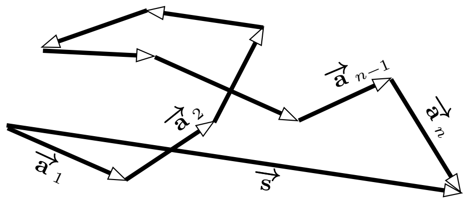


Figure 3: The vector sum of n vectors.

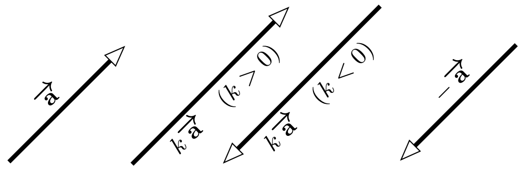


Figure 4: Multiplication of a vector by a scalar and the negative of a vector.

commutative and *associative*. That is for three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have

$$\vec{a}+\vec{b} = \vec{b}+\vec{a} \qquad \text{(Commutativity)} \tag{1}$$

$$\left(\vec{a}+\vec{b}\right)+\vec{c} = \vec{a}+\left(\vec{b}+\vec{c}\right) \qquad \text{(Associativity)} \tag{2}$$

Thus we can add vectors in any order and there is no need to put brackets while adding more than two vectors.

1.1.2 Multiplication by a scalar

Suppose k is a non–zero scalar quantity and $\vec{a}$ is a vector. The multiplication of $\vec{a}$ by the scalar k:

$$k\vec{a}$$

results in a vector whose

- *magnitude* is equal to the magnitude of $\vec{a}$ multiplied by the absolute value of k,

$$|k\vec{a}| = |k||\vec{a}| \tag{3}$$

- *direction* is same as that of $\vec{a}$ if k is positive and opposite that of $\vec{a}$ if k is negative (Figure 4).

If a vector $\vec{\mathbf{a}}$ is multiplied by $k = -1$, we obtain the *negative* (or *additive inverse*) as $-\vec{\mathbf{a}}$ as a vector whose magnitude is equal to that of $\vec{\mathbf{a}}$ while its direction is exactly opposite to that of $\vec{\mathbf{a}}$.

Making use of the above defined multiplication we can also *subtract* a vector $\vec{\mathbf{b}}$ from $\vec{\mathbf{a}}$ as follows:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}})$$

That is, to subtract $\vec{\mathbf{b}}$ from $\vec{\mathbf{a}}$, just add the negative of $\vec{\mathbf{b}}$ to $\vec{\mathbf{a}}$.

Finally, note that the scalar multiplication is *distributive* over vector addition:

$$k\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) = k\,\vec{\mathbf{a}} + k\,\vec{\mathbf{b}} \tag{4}$$

1.1.3 Dot product of two vectors

The *dot product* of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is defined by

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \equiv |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta = ab\cos\theta \tag{5}$$

where θ is the angle they form when placed tail–to–tail (Figure 5). Note that $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ itself is a

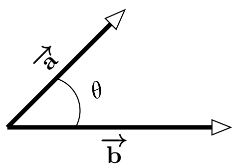
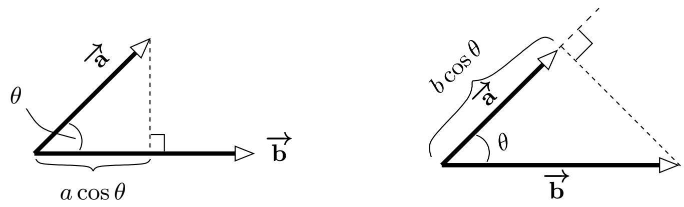


Figure 5: Dot product of two vectors, the angle between them being θ.



(a) (b)

Figure 6: Two different geometrical interpretation of the dot product.

scalar quantity (hence the alternate name *scalar product*). The dot product is *commutative*,

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}} \tag{6}$$

and *distributive over vector addition*,

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} \tag{7}$$

Geometrically, the dot product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is the product of the magnitude of $\vec{\mathbf{a}}$ and the *projection* of $\vec{\mathbf{b}}$ along the direction of $\vec{\mathbf{a}}$ (or the product of the magnitude of $\vec{\mathbf{b}}$ and the projection of $\vec{\mathbf{a}}$ along the direction of $\vec{\mathbf{b}}$) (Figure 6).

If the two vectors are parallel, then

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| = ab$$

In particular, for any vector $\vec{\mathbf{a}}$, we obtain

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = |\vec{\mathbf{a}}|^2 = a^2 \qquad \Rightarrow \qquad a = \sqrt{\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}} \tag{8}$$

If the two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are perpendicular to each other, then

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$$

♣ *Remark* 1. A *unit vector* is a vector whose magnitude is one. Thus, it only indicates a direction. If we have a vector $\vec{\mathbf{r}}$, then the unit vector $\hat{\mathbf{r}}$ (read as "r–hat" or "r–cap") in its direction can be obtained by dividing the vector $\vec{\mathbf{r}}$ by its magnitude:

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \frac{\vec{\mathbf{r}}}{r}$$

Since the unit vector $\hat{\mathbf{r}}$ is the ratio of two quantities that have the same dimensions, the unit vector is a dimensionless quantity.

1.1.4 Cross product of two vectors

The *cross* or *vector* product of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is defined by

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \equiv |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta \ \hat{\mathbf{n}} = ab \sin\theta \ \hat{\mathbf{n}} \tag{9}$$

where θ is the angle between the two vectors when drawn from the same point and $\hat{\mathbf{n}}$ is a unit vector along a direction that is perpendicular to the plane formed by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (see Figure 7). Of course, there are *two* directions perpendicular to any plane: "in" and "out". The

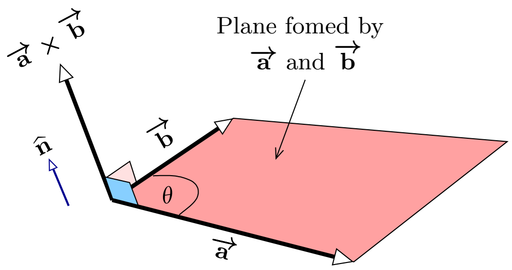


Figure 7: The cross product of two vectors.

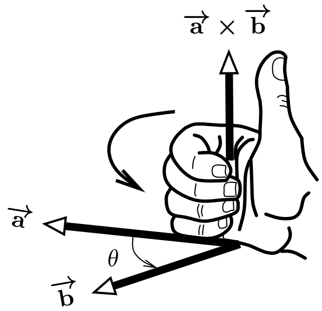


Figure 8: Illustrating the righthand thumb rule.

ambiguity is resolved by the **right hand thumb rule**: let your fingers point in the direction of the first vector and then curl them (via the smaller angle) towards the second vector (Figure 8); then your thumb indicates the direction of $\hat{\mathbf{n}}$ and hence also of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

Note that $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is itself a *vector* (hence the alternative name *vector product*). Also note that the cross product is *not* commutative, that is

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \neq \vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

In fact,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\left(\vec{\mathbf{b}} \times \vec{\mathbf{a}}\right) \tag{10}$$

But the cross product is distributive over vector addition:

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \times \vec{\mathbf{c}}) \tag{11}$$

Geometrically, the *magnitude* of the cross product, $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$, is the area of the parallelogram generated by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ when placed tail–to–tail. If two vectors are parallel (or antiparallel), then their cross product is zero. In particular, for any vector $\vec{\mathbf{a}}$, we have

$$\vec{\mathbf{a}} \times \vec{\mathbf{a}} = 0$$

1.2 Vector algebra: component form

So far I have defined the four vector operations (addition, scalar multiplication, dot product, and cross product) in "abstract" form — that is, without reference to any particular coordinate system. In practice, it is often convenient to set up a Cartesian coordinate system and work with the "components" of the vector.

A *set of mutually perpendicular directions together with an origin and a unit for measurement of length* constitutes a rectangular Cartesian coordinate system. Once the directions are chosen, we arbitrarily denote them[2] as the x, y, and z axis and express the direction in terms of the unit vectors: $\widehat{\mathbf{i}}$ along the $+x$ direction; $\widehat{\mathbf{j}}$ along the $+y$ direction; $\widehat{\mathbf{k}}$ along the $+z$ direction (see Figure 9).

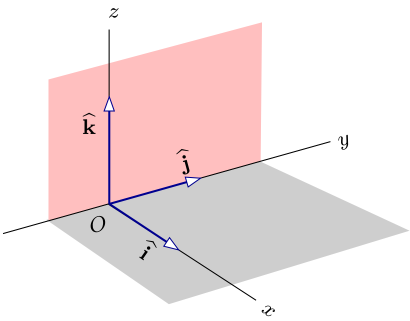


Figure 9: A coordinate system and the unit vectors.

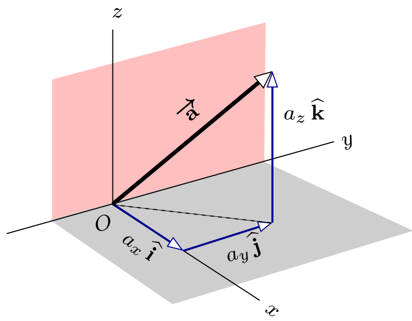


Figure 10: The components of a vector.

Consider a vector $\overrightarrow{\mathbf{a}}$ that has been drawn so as to originate from the origin O of the coordinate system (Figure 10). As has been shown, the vector $\overrightarrow{\mathbf{a}}$ can be considered as the sum of three vectors: $a_x\,\widehat{\mathbf{i}}$, $a_y\,\widehat{\mathbf{j}}$, and $a_z\,\widehat{\mathbf{k}}$. What exactly is the meaning of $a_x\,\widehat{\mathbf{i}}$? It simply means that we have to measure a_x unit from the origin along the $\widehat{\mathbf{i}}$ direction. If $a_x > 0$, the measurement has to be done along the positive direction of $\widehat{\mathbf{i}}$ while if $a_x < 0$, the measurement has to be taken along the opposite direction. Similar meaning hold for the other two directions. Thus, we can write (using vector sum):

$$\overrightarrow{\mathbf{a}} = a_x\,\widehat{\mathbf{i}} + a_y\,\widehat{\mathbf{j}} + a_z\,\widehat{\mathbf{k}} \tag{12}$$

In the above equation, the *scalars* a_x, a_y, and a_z are the *components* of the vector $\overrightarrow{\mathbf{a}}$ along the x, y and z directions respectively. Geometrically, they are the projections of $\overrightarrow{\mathbf{a}}$ along the three coordinate axes. Keep in mind that the components of a vector are scalars, as such they can be positive, negative or even zero.

We can now reformulate each of the four vector operations as a rule for manipulating components.

1. Addition. For two vectors $\overrightarrow{\mathbf{a}} = a_x\,\widehat{\mathbf{i}} + a_y\,\widehat{\mathbf{j}} + a_z\,\widehat{\mathbf{k}}$ and $\overrightarrow{\mathbf{b}} = b_x\,\widehat{\mathbf{i}} + b_y\,\widehat{\mathbf{j}} + b_z\,\widehat{\mathbf{k}}$, we have

$$\begin{aligned}\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} &= \left(a_x\,\widehat{\mathbf{i}} + a_y\,\widehat{\mathbf{j}} + a_z\,\widehat{\mathbf{k}}\right) + \left(b_x\,\widehat{\mathbf{i}} + b_y\,\widehat{\mathbf{j}} + b_z\,\widehat{\mathbf{k}}\right) \\ &= (a_x + b_x)\,\widehat{\mathbf{i}} + (a_y + b_y)\,\widehat{\mathbf{j}} + (a_z + b_z)\,\widehat{\mathbf{k}}\end{aligned} \tag{13}$$

Thus, *to add vectors, add the like components.*

2. Scalar multiplication. For the vector $\overrightarrow{\mathbf{a}} = a_x\,\widehat{\mathbf{i}} + a_y\,\widehat{\mathbf{j}} + a_z\,\widehat{\mathbf{k}}$ and a scalar number k,

$$k\,\overrightarrow{\mathbf{a}} = (ka_x)\,\widehat{\mathbf{i}} + (ka_y)\,\widehat{\mathbf{j}} + (ka_z)\,\widehat{\mathbf{k}} \tag{14}$$

[2] We generally choose the directions so that they form a "right handed system".

Thus, *to multiply a vector by a scalar, multiply each component by that scalar.*

3. Dot product. Because $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$ and $\widehat{\mathbf{k}}$ are mutually perpendicular to each other,

$$\widehat{\mathbf{i}}\cdot\widehat{\mathbf{i}}=\widehat{\mathbf{j}}\cdot\widehat{\mathbf{j}}=\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}=1; \qquad \widehat{\mathbf{i}}\cdot\widehat{\mathbf{j}}=\widehat{\mathbf{i}}\cdot\widehat{\mathbf{k}}=\widehat{\mathbf{j}}\cdot\widehat{\mathbf{k}}=0 \tag{15}$$

Accordingly,

$$\begin{aligned}\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} &= \left(a_x\,\widehat{\mathbf{i}}+a_y\,\widehat{\mathbf{j}}+a_z\,\widehat{\mathbf{k}}\right)\cdot\left(b_x\,\widehat{\mathbf{i}}+b_y\,\widehat{\mathbf{j}}+b_z\,\widehat{\mathbf{k}}\right)\\ &= a_xb_x+a_yb_y+a_zb_z\end{aligned} \tag{16}$$

Thus, *to calculate the dot product, multiply like components and add.*

In particular,

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}=a_x^2+a_y^2+a_z^2 \qquad \Rightarrow \quad |\vec{\mathbf{a}}|=a=\sqrt{a_x^2+a_y^2+a_z^2} \tag{17}$$

This is, if you like, the three–dimensional generalization of the Pythagoras theorem. Note that the dot product of $\vec{\mathbf{a}}$ with any *unit* vector is the component of $\vec{\mathbf{a}}$ along that direction. Thus

$$\vec{\mathbf{a}}\cdot\widehat{\mathbf{i}}=a_x; \qquad \vec{\mathbf{a}}\cdot\widehat{\mathbf{j}}=a_y; \qquad \vec{\mathbf{a}}\cdot\widehat{\mathbf{k}}=a_z$$

Further, writing the left hand side of (16) as $ab\cos\theta$, where θ is the angle between the two vectors, we obtain

$$ab\cos\theta=a_xb_x+a_yb_y+a_zb_z \qquad \Rightarrow \quad \cos\theta=\frac{a_xb_x+a_yb_y+a_zb_z}{ab}$$

Thus, the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given by

$$\cos\theta=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\,\sqrt{b_x^2+b_y^2+b_z^2}} \tag{18}$$

Since, for any value of θ, $0\le\cos^2\theta\le 1$, we obtain an important result known as the *Cauchy–Schwartz inequality*:

$$\left(a_xb_x+a_yb_y+a_zb_z\right)^2\le\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right) \tag{19}$$

4. Cross product. For the unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$, it can be easily verified that

$$\begin{aligned}\widehat{\mathbf{i}}\times\widehat{\mathbf{i}} &= \quad\widehat{\mathbf{j}}\times\widehat{\mathbf{j}} &&=\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}=0\\ \widehat{\mathbf{i}}\times\widehat{\mathbf{j}} &= -(\widehat{\mathbf{j}}\times\widehat{\mathbf{i}}) &&=\widehat{\mathbf{k}}\\ \widehat{\mathbf{j}}\times\widehat{\mathbf{k}} &= -(\widehat{\mathbf{k}}\times\widehat{\mathbf{j}}) &&=\widehat{\mathbf{i}}\\ \widehat{\mathbf{k}}\times\widehat{\mathbf{i}} &= -(\widehat{\mathbf{i}}\times\widehat{\mathbf{k}}) &&=\widehat{\mathbf{j}}\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}\vec{\mathbf{a}}\times\vec{\mathbf{b}} &= \left(a_x\,\widehat{\mathbf{i}}+a_y\,\widehat{\mathbf{j}}+a_z\,\widehat{\mathbf{k}}\right)\times\left(b_x\,\widehat{\mathbf{i}}+b_y\,\widehat{\mathbf{j}}+b_z\,\widehat{\mathbf{k}}\right)\\ &= (a_yb_z-a_zb_y)\widehat{\mathbf{i}}+(a_zb_x-a_xb_z)\widehat{\mathbf{j}}+(a_xb_y-a_yb_x)\widehat{\mathbf{k}}\end{aligned} \tag{21}$$

This cumbersome expression can be written more neatly as a *determinant*:

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\begin{vmatrix}\widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \tag{22}$$

Thus, *to calculate the cross product of* $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, *form a determinant whose first row is* $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$, *whose second row is* $\vec{\mathbf{a}}$ *(in component form), and whose third row is* $\vec{\mathbf{b}}$ *(again in component form).*