Assignment P–11–2

(Vectors)

1. (a) How many vectors can be perpendicular to a given vector? (b) How many vectors can be perpendicular to a given vector, if they all lie in the same plane as that of the given vector?

2. Can the sum of three unit vectors be a unit vector?

3. The sum of the magnitudes of two vectors is 18 units and the magnitude of their resultant is 12 units. The resultant is at 90° with the vector having the smaller magnitude. Find the magnitude of the resultant vectors.

4. Find a vector of 15 units in the direction of the vector $3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$.

5. Find the angle between the vectors $2\,\hat{\mathbf{i}}$ and $3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$.

6. The vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ have the same magnitude. Find the angle between the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

7. Consider a vector joining the origin to the point (1, 1, 1). If the sun's rays are in the negative y direction, find the projection of the vector on the xz plane.

8. Let $\vec{\mathbf{a}} = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\vec{\mathbf{b}} = 3\,\hat{\mathbf{i}}$, and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$, where $\hat{\mathbf{i}}$, and $\hat{\mathbf{j}}$ are unit vectors along the positive directions of the x and y axes. Perform the following operations graphically and analytically: (i) $\vec{\mathbf{a}} + \vec{\mathbf{b}} - \vec{\mathbf{c}}$, (ii) $2\,\vec{\mathbf{b}} + \vec{\mathbf{c}}$, (iii) $3\,\vec{\mathbf{c}} - \vec{\mathbf{a}}$.
In each case find the magnitude of the resulting vector and the angle that it makes with the positive x direction.

9. Vector $\vec{\mathbf{a}}$ has a magnitude of 8.00 units and makes an angle of 45.0° with the positive x axis. Vector $\vec{\mathbf{b}}$ also has a magnitude of 8.00 units and is directed along the negative x axis. Find the magnitude and the angle that is made with the x axis by the following vectors: (i) the vector sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and (ii) the vector difference $\vec{\mathbf{a}} - \vec{\mathbf{b}}$.

10. An aeroplane starting from airport A flies 300 km east, then 350 km at 30.0° west of north, and then 150 km north to arrive finally at airport B. (a) The next day, another plane flies directly from airport A to airport B in a straight line. In what direction should the pilot travel in this direct flight? (b) How far will the pilot travel in this direct flight? Assume there is no wind during these flights.

11. If $\vec{\mathbf{a}} = (6.00\,\hat{\mathbf{i}} - 8.00\,\hat{\mathbf{j}})$ units, $\vec{\mathbf{b}} = (-8.00\,\hat{\mathbf{i}} + 3.00\,\hat{\mathbf{j}})$ units, and $\vec{\mathbf{c}} = (26.0\,\hat{\mathbf{i}} + 19.0\,\hat{\mathbf{j}})$ units, determine scalars ℓ and m such that $\ell\,\vec{\mathbf{a}} + m\,\vec{\mathbf{b}} + \vec{\mathbf{c}} = 0$.

12. Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have exactly the same magnitudes. What must be the angle between them, if the magnitude of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is n times that of $\vec{\mathbf{a}} - \vec{\mathbf{b}}$?

13. Find the sum of these four vectors: 12.0 units to the right at 35.0° above the horizontal, 31.0 units to the left at 55.0° above the horizontal, 8.40 units to the left at 35.0° below the horizontal, and 24.0 units to the right at 55.0° below the horizontal. (Hint: Make a drawing of this situation and select the best axes for x and y so that you have the least number of components. Then add the vectors, using the component method.)

14. A pirate has buried his treasure on an island with five trees located at the following points: A(30.0 m, −20.0 m), B(60.0 m, 80.0 m), C(−10.0 m, −10.0 m), D(40.0 m, −30.0 m), and E(−70.0 m, 60.0 m). All points are measured relative to some origin, as in Figure 1. Instructions on the map tell you to start at A and move toward B, but to cover only one–half the distance between A and B. Then, move toward C, covering one–third the distance between your current location and C. Next,

move toward D, covering one–fourth the distance between where you are and D. Finally, move toward E, covering one–fifth the distance between you and E, stop, and dig. (a) What are the coordinates of the point where the pirate's treasure is buried? (b) Rearrange the order of the trees, (for instance, B(30.0 m, −20.0 m), A(60.0 m, 80.0 m), E(−10.0 m, −10.0 m), C(40.0 m, −30.0 m), and D(−70.0 m, 60.0 m), and repeat the calculation to show that the answer does not depend on the order of the trees.

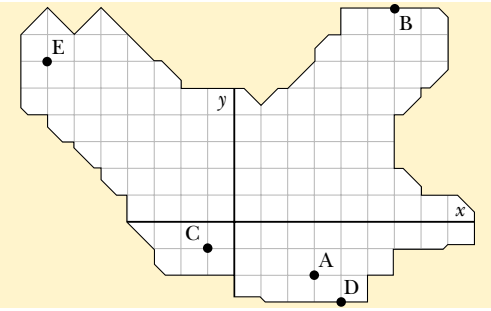

Figure 1: Treasure hunt.

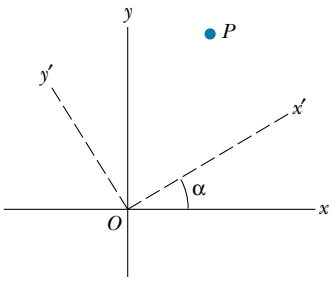

Figure 2:

15. A point P is described by the vector $a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$ with respect to the normal Cartesian coordinate system shown in Figure 2. Now the coordinate system is rotated by an angle α as shown. Find the components new a'_x and a'_y of the vector along the new directions x' and y'.

16. A bird initially at the origin starts flying in a direction such that it makes an angle 30° with the xy plane and it is always equidistant from the x and y axis. Consider the z axis as representing the height. At the same instant when the bird starts flying, a car starts moving from the origin such that it is always vertically below the car. If the bird is flying at a speed of 10 m/s, find the direction of motion and the speed of the car.

17. Prove that for a fixed natural number n, the following hold true:

$$\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0, \qquad \text{and} \qquad \sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$$

18. If $\vec{\mathbf{a}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = 3\hat{\mathbf{j}}$, find a vector of magnitude 10 units in the direction of the vector $2\vec{\mathbf{a}} + 3\vec{\mathbf{b}}$.

19. If $\vec{\mathbf{a}}$ is a unit vector and $(\vec{\mathbf{x}} - \vec{\mathbf{a}}) \cdot (\vec{\mathbf{x}} + \vec{\mathbf{a}}) = 8$, then find $|\vec{\mathbf{x}}|$.

20. If $\vec{\mathbf{a}} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, then find the angle between the vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$.

21. If $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ are unit vectors such that $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$, then find the value of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{c}} \cdot \vec{\mathbf{a}}$.

22. If $|\vec{\mathbf{a}}| = 3$, $|\vec{\mathbf{b}}| = 4$, and $|\vec{\mathbf{c}}| = 5$ such that each vector is perpendicular to the sum of the other two, find $|\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}|$.

23. Find the cross product of the following pairs of vectors: (i) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$; (ii) $\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

24. Find unit vectors perpendicular to each of the vectors $(\vec{\mathbf{u}} + \vec{\mathbf{v}})$ and $(\vec{\mathbf{u}} - \vec{\mathbf{v}})$, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

25. Find λ and μ if $(2\hat{\mathbf{i}}+6\hat{\mathbf{j}}+27\hat{\mathbf{k}})\times(\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\mu\hat{\mathbf{k}})=\vec{\mathbf{0}}$

26. Given that $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=0$ and $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}}$, what can you conclude about vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$?

27. Find the angle between the two body diagonals of a cube.

28. Prove that the vectors $|\vec{\mathbf{u}}|\vec{\mathbf{v}}+|\vec{\mathbf{v}}|\vec{\mathbf{u}}$ and $|\vec{\mathbf{u}}|\vec{\mathbf{v}}-|\vec{\mathbf{v}}|\vec{\mathbf{u}}$ are perpendicular for any vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

29. If $\vec{\mathbf{a}}\neq\vec{\mathbf{0}}$, do $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$ and $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{a}}\times\vec{\mathbf{c}}$ together imply $\vec{\mathbf{b}}=\vec{\mathbf{c}}$?

30. A vector, whose magnitude is $5\sqrt{6}$ units, is directed along the internal bisector of the angle between the vectors $7\hat{\mathbf{i}}-4\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ and $-2\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. Find the vector.

31. A vector $\vec{\mathbf{a}}$ has components $2p$ and 1 with respect to a rectangular Cartesian system. The coordinate system is now rotated through a certain angle in the anticlockwise sense keeping the origin fixed. If $\vec{\mathbf{a}}$ has components $p+1$ and 1 with respect to the new system, then find p.

32. If $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ are unit vectors, then show that

$$0\leq\left|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right|^2+\left|\vec{\mathbf{b}}-\vec{\mathbf{c}}\right|^2+\left|\vec{\mathbf{c}}-\vec{\mathbf{a}}\right|^2\leq 9$$

and find the condition for which the upper bound is attained.

33. The vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ are of the same length and, taken in pair, they form equal angles. If $\vec{\mathbf{a}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\mathbf{b}}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$, find $\vec{\mathbf{c}}$.