Solutions

(Selected problems from Assignment on Continuity)

1. Find all points of continuity of f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \sin|x| & \text{if } x \text{ is rational.} \end{cases}$$

Solution: The function is discontinuous at each $x_0 \neq k\pi$, where $k \in \mathbb{Z}$. Indeed, if $\{x_n\}$ is a sequence of irrationals converging to x_0, then the corresponding sequence of the function values $\{f(x_n)\} = \{0\}$ and hence $\{f(x_n)\}$ converges to 0. On the other hand if $\{x_n^*\}$ is a sequence of rational numbers converging to x_0, then the corresponding sequence of the function values $\{f(x_n^*)\} = \{\sin|x_n^*|\}$ which converges to $\sin|x_0| \neq 0$ (by the continuity of sine function). Similarly, one can show that f is continuous at all points of the form $k\pi$, with $k \in \mathbb{Z}$. □

2. Determine the set of points of continuity of f defined by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Solution: As in the previous solution, it can be shown that f is continuous only at -1 and 1. □

3. Study the continuity of:
(a) The *Riemann function* defined as

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0, \\ \dfrac{1}{q} & \text{if } x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{N}, \\ & \text{and } p, q \text{ are co-prime.} \end{cases}$$

Solution: First notice that if a sequence $\{x_n\}$, where $x_n = \dfrac{p_n}{q_n}$ with $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{N}$ being relatively prime, converges to x (with $x_n \neq x$ for any n), then $\lim\limits_{n\to\infty} q_n = \infty$. So if x is irrational and $\{x_n\}$ is as above, then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} \dfrac{1}{q_n} = 0 = f(x)$. On the other hand, if $\{z_n\}$ is a sequence of irrationals converging to x, then $\lim\limits_{n\to\infty} f(z_n) = 0 = f(x)$. This means that f is continuous at every irrationals. Similarly, it can be proved that f is also continuous at 0.

Suppose now that $x \neq 0$ and $x = \dfrac{p}{q}$, where p and q are co-prime. If $\{x_n\}$ is a sequence of irrationals converging to x, then $\lim\limits_{n\to\infty} f(x_n) = 0 \neq f(x)$. Consequently, f is discontinuous at every rational number other than 0. □

(b) The function defined as

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is irrational or } x = 0, \\ \dfrac{qx}{q+1} & \text{if } x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{N}, \\ & \text{and } p, q \text{ are co-prime.} \end{cases}$$

Solution: As in part (a), it can be shown that f is discontinuous at all rational numbers except 0. □

4. Prove that if a function f is continuous on $[a, b]$,the $|f|$ is also continuous on the given interval. Show by an example that the converse in not true.

Solution: Consider an arbitrary point $x_0 \in [a, b]$. Continuity of f at x_0 means

$$\lim_{x\to x_0} f(x) = f(x_0) \quad \Rightarrow \lim_{x\to x_0} (f(x) - f(x_0)) = 0$$

Since $||a| - |b|| \leq |a - b|$, we get

$$\big||f(x)| - |f(x_0)|\big| \leq |f(x) - f(x_0)|$$

which gives

$$-|f(x) - f(x_0)| \leq |f(x)| - |f(x_0)| \leq |f(x) - f(x_0)|$$

Passing on to the limit and using the Sandwich theorem, we get

$$\lim_{x\to x_0} (|f(x)| - |f(x_0)|) = 0 \quad \Rightarrow \lim_{x\to x_0} |f(x)| = |f(x_0)|$$

which means that the function $|f(x)|$ is also continuous on $[a, b]$.

The function given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational number} \in [a, b], \\ -1 & \text{if } x \text{ is irrational number } \in [a, b]. \end{cases}$$

is discontinuous at each point in $[a, b]$, although $|f|$ is a constant function and therefore continuous on $[a, b]$. □

5. Determine all a_n and b_n for which the function defined by

$$f(x) = \begin{cases} a_n + \sin \pi x & \text{if } x \in [2n, 2n+1],\ n \in \mathbb{Z}, \\ b_n + \cos \pi x & \text{if } x \in (2n-1, 2n),\ n \in \mathbb{Z}. \end{cases}$$

is continuous on $\mathbb{R}$.

Answer: $a_n = 2n + a_0$ and $b_n = 2n - 1 + a_0$, where $a_0 \in \mathbb{R}$. □

6. Let $f(x) = [x^2] \sin \pi x$ for $x \in \mathbb{R}$, where [.] denotes the greatest integer function. Study the continuity of f.

Answer: f is continuous at all real numbers except for those which are of the form $x = \pm\sqrt{n}$ where n is an integer that is not a perfect square. □

7. Let f be defined as

$$f(x) = [x] + (x - [x])^{[x]} \quad \text{for } x \geq \frac{1}{2}$$

Show that f is continuous and that it is strictly increasing on $[1, \infty)$.

Solution: We have

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{2}, 1\right), \\ n + (x-n)^n & \text{if } x \in [n, n+1),\ n \in \mathbb{N}. \end{cases}$$

Consequently, the function is continuous at each $x \neq n$, $n \in \mathbb{N}$. Also,

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^-} f(x) = n = f(n).$$

So, f is continuous at all $x \in [1/2, \infty)$.

Next, we prove that f is strictly increasing of $[1, \infty)$. It is quite clearly, that in each of the intervals $[n, n+1)$, f is increasing. If $x_1 \in [n-1, n)$ and $x_2 \in [n, n+1)$, then

$$\begin{aligned} f(x_2) - f(x_1) &= (x_2 - n)^n + 1 - (x_1 - n + 1)^{n-1} \\ &> (x_2 - n)^n \\ &\geq 0 \end{aligned}$$

It then follows that $f(x_2) - f(x_1) > 0$ for $x_2 \in [m, m+1)$ and $x_1 \in [n, n+1)$, if $m > n+1$. As such f is strictly increasing for $x \in [1, \infty)$. ☐

8. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then it attains a maximum and a minimum value.

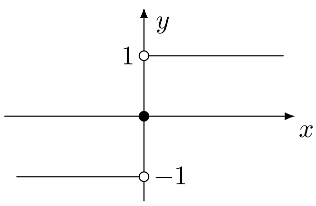
Solution: The required result follows by application of extreme value principle to the function in the interval $[0, T]$, where $T > 0$ is the period of f. ☐

9. Study the continuity of the following functions and sketch their graphs:

(a) $f(x) = \lim\limits_{n\to\infty} \dfrac{n^x - n^{-x}}{n^x + n^{-x}}$ for $x \in \mathbb{R}$

The given function is

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

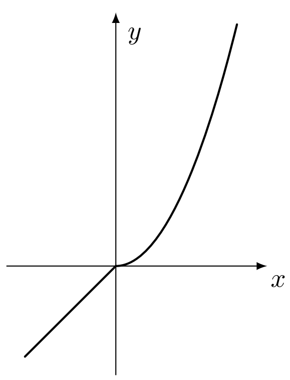


As such it is discontinuous at $x = 0$.

(b) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^2 e^{nx} + x}{e^{nx} + 1}$ for $x \in \mathbb{R}$

By its definition

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$



This function is continuous for all x.

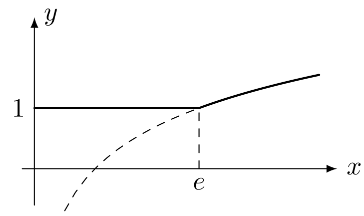
(c) $f(x) = \lim\limits_{n\to\infty} \dfrac{\ln(e^n + x^n)}{n}$ for $x \geq 0$

We get

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \frac{\ln(e^n + x^n)}{n} \\ &= \lim_{n\to\infty} \frac{n + \ln(1 + (x/e)^n)}{n} \end{aligned}$$

Consequently,

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq e, \\ \ln x & \text{if } x > e. \end{cases}$$

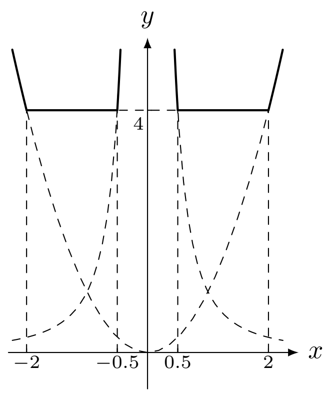


The function is continuous on $[0, \infty)$.

(d) $f(x) = \lim\limits_{n\to+\infty} \sqrt[n]{4^n + x^{2n} + \dfrac{1}{x^{2n}}}$ where $x \neq 0$

It is easy to verify that

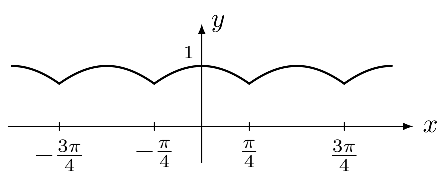
$$f(x) = \max\left\{4, x^2, \frac{1}{x^2}\right\}$$



It is continuous on $\mathbb{R} - \{0\}$.

(e) $f(x) = \lim\limits_{n\to\infty} \sqrt[2n]{\cos^{2n} x + \sin^{2n} x}$ for $x \in \mathbb{R}$

The given function is $f(x) = \max\{|\cos x|, |\sin x|\}$.



Clearly, f is continuous on $\mathbb{R}$.

10. Give an example of a bounded function on $[0, 1]$ which achieves neither a minimum nor a maximum.

Answer: There are infinitely many such functions. For e.g.

$$f(x) = \begin{cases} 2x-1 & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

□

11. Give an example of a bounded function on $[0,1]$ which does not achieve its minimum on any $[a,b] \subset [0,1]$, where $a < b$.

Solution: For $n \in \mathbb{N}$, define

$$A_n = \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \ldots, \frac{2^n-1}{2^n}\right\}$$

and $B_1 = A_1$, $B_n = A_n - \bigcup_{k=1}^{n-1} A_k = A_n - A_{n-1}$. Clearly

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$$

Define f as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) - \bigcup_{k=1}^{\infty} A_k, \\ \dfrac{1}{2^n} - 1 & \text{if } x \in B_n, n \in \mathbb{N}. \end{cases}$$

For any a and b, where $0 \le a < b \le 1$, there is no minimum of f on $[a,b]$ although it can reach arbitrarily close to -1. □

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with two *incommensurable* periods T_1 and T_2; that is $\frac{T_1}{T_2}$ is irrational. Prove that f is a constant function. Give an example of a non-constant periodic function with two incommensurable periods.

Solution: First, notice that the set

$$\left\{m + n\frac{T_1}{T_2} \,|\, m, n \in \mathbb{Z}\right\}$$

is infinitely dense, i.e. given any $x \in \mathbb{R}$, there is a sequence $\left\{m_k + n_k \frac{T_1}{T_2}\right\}$ convergent to $\dfrac{x}{T_2}$. Therefore, by the periodicity and continuity of f, we get

$$f(0) = \lim_{k\to\infty} f(m_k T_2 + n_k T_1) = f(x)$$

so that f is constant.

For the second part, let T_1 and T_2 be two incommensurable numbers and let

$$X = \{x \in \mathbb{R} \,|\, x = rT_1 + sT_2,\ r, s \in \mathbb{Q}\}.$$

Define the function $\psi(x)$ as

$$\psi(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \in \mathbb{R} - X. \end{cases}$$

Then T_1 and T_2 are periods of $\psi(x)$. □

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with positive fundamental periods T_1 and T_2, respectively. Prove that if $\frac{T_1}{T_2} \notin \mathbb{Q}$, then $h = f + g$ is not a periodic function.

Solution: Suppose on the contrary that $h = f + g$ is periodic with period T. Since $\frac{T_1}{T_2} \notin \mathbb{Q}$, so either $\frac{T}{T_1} \notin \mathbb{Q}$ or $\frac{T}{T_2} \notin \mathbb{Q}$. Assume for example, that $\frac{T}{T_1} \notin \mathbb{Q}$. By periodicity of h we get $f(x+T) + g(x+T) = h(x+T) = h(x) = f(x) + g(x)$ for $x \in \mathbb{R}$. Therefore the function H defined by setting $H(x) = f(x+T) - f(x) = g(x) - g(x+T)$ is continuous and periodic (and hence bounded; see Problem 8) with two incommensurable periods T_1 and T_2. Therefore, by the result of the previous problem, $H(x)$ is constant. This means that there exists some C so that $f(x+T) = f(x) + C$ for $x \in \mathbb{R}$. If $C \neq 0$, then setting $x = 0$ and then $x = T$ in the last equality, we get

$$f(2T) = f(T) + C = f(0) + 2C$$

Similarly

$$f(3T) = f(2T) + C = f(0) + 3C$$

and so on. It can be shown inductively, that $f(nT) = f(0) + nC$, which contradicts the boundedness of f. Hence, $C = 0$. So we have $f(x+T) = f(x)$ meaning that T is a period of f as well. But this means that $\frac{T}{T_1}$ is a rational number, which is a contradiction. □

14. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a *fixed point* in $[0,1]$; that is, there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

Hint: Consider that function $g(x) = f(x) - x$ for $x \in [0,1]$. □

15. Assume that $f, g : [a,b] \to \mathbb{R}$ are continuous and such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $x_0 \in (a,b)$ for which $f(x_0) = g(x_0)$.

Solution: Let $h(x) = f(x) - g(x)$ for $x \in [a,b]$. Obviously, h is continuous. Further, $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. Since $h(a)$ and $h(b)$ are of opposite sign, h must attain the value of zero atleast once. That is, there is a $x_0 \in (a,b)$, so that $h(x_0) = 0$, i.e. $f(x_0) = g(x_0)$. □

16. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that

$$f\left(x_0 + \frac{T}{2}\right) = f(x_0)$$

Solution: Define $g(x) = f\left(x + \frac{T}{2}\right) - f(x)$ for $x \in \left[0, \frac{T}{2}\right]$. Then $g(0) = f\left(\frac{T}{2}\right) - f(0)$ while

$$g\left(\frac{T}{2}\right) = f(T) - f\left(\frac{T}{2}\right) = f(0) - f\left(\frac{T}{2}\right) = -g(0)$$

Since g is continuous, there exists some $x_0 \in [0, T/2]$ such that $g(x_0) = 0$; that is to say

$$f\left(x_0 + \frac{T}{2}\right) = f(x_0)$$

□

17. A function $f : (a, b) \to \mathbb{R}$ is continuous. Prove that, given $x_1, x_2, \ldots, x_n$ in (a, b), there exists $x_0 \in (a, b)$, such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$$

Solution: Let $m = \min\{f(x_1), f(x_2), \ldots, f(x_n)\}$ and $M = \max\{f(x_1), f(x_2), \ldots, f(x_n)\}$. Then

$$m \leq \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \leq M$$

Consequently, there is $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$$

□

18. (a) Prove that the equation $(1 - x)\cos x = \sin x$ has at least one solution in (0, 1).

Solution: Set $f(x) = (1 - x)\cos x - \sin x$. Obviously f is continuous. Also $f(0) = 1$ and $f(1) = -\sin 1 < 0$. Therefore, there is $x_0 \in (0, 1)$ satisfying $f(x_0) = 0$. □

(b) For a nonzero polynomial $P(x)$, show that the equation $|P(x)| = e^x$ has at least one real solution.

Solution: Clearly $e^{-x}|P(x)| - 1$ is continuous. Further, it can be easily seen that

$$\lim_{x\to\infty}(e^{-x}|P(x)| - 1) = -1 \quad \text{and} \quad \lim_{x\to\infty} e^{-x}|P(x)| = +\infty$$

Consequently, there is an $x_0 \in \mathbb{R}$ such that $e^{-x_0}|P(x_0)| - 1 = 0$ i.e. $|P(x)| = e^{x_0}$. □

19. For $a_0 < b_0 < a_1 < b_1 < \ldots < a_n < b_n$, show that all the roots of the polynomial

$$P(x) = \prod_{k=0}^{n}(x + a_k) + 2\prod_{k=0}^{n}(x + b_k), \quad x \in \mathbb{R}$$

are real.

Solution: Observe that

$$\begin{aligned} \text{sign } P(-a_k) &= (-1)^k, \\ \text{and sign } P(-b_k) &= (-1)^{k+1},\ k = 0, 1, 2, \ldots, n \end{aligned}$$

By the intermediate value property, there is a root of the polynomial P in every interval $(-b_k, -a_k)$, for $k = 0, 1, 2, \ldots, n$. Being a polynomial of degree $n + 1$, P can have at max $n + 1$ real roots. Hence all roots of P are real. □

20. Suppose that f and g have the intermediate value property on $[a, b]$. Must $f + g$ posses the intermediate value property on that interval?

Answer: No. Consider, for example, f and g defined as follows:

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x - a}\right) & \text{if } a < x \leq b, \\ 0 & \text{if } x = a, \end{cases}$$

and

$$g(x) = \begin{cases} -\sin\left(\dfrac{1}{x - a}\right) & \text{if } a < x \leq b, \\ 0 & \text{if } x = a. \end{cases}$$

□

21. Assume that f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0, 2]$ such that $x_2 - x_2 = 1$ and $f(x_2) = f(x_1)$. Give a geometric interpretation of this fact.

Solution: Define another function

$$g(x) = f(x + 1) - f(x), \quad \text{for } x \in [0, 1]$$

Then g is continuous and additionally $g(1) = f(2) - f(1) = -g(0)$. Hence, there exists $x_0 \in [0, 1]$ such that $g(x_0) = 0$, meaning $f(x_0 + 1) = f(x_0)$. So we can take, $x_1 = x_0$ and $x_2 = x_0 + 1$. □

22. Let f be continuous on $[0, 2]$. Show that there are x_1 and x_2 in $[0, 2]$ such that $x_2 - x_1 = 1$ and $f(x_2) - f(x_1) = \dfrac{f(2) - f(0)}{2}$.

Solution: Define a function

$$g(x) = f(x + 1) - f(x) - \frac{f(2) - f(0)}{2}, \quad \text{for } x \in [0, 1]$$

and proceed as in the previous problem. □

23. For $n \in \mathbb{N}$, let f be continuous on $[0, n]$ such that $f(0) = f(n)$. Prove that there exist x_1 and x_2 in $[0, n]$ satisfying $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

Solution: Define the function g by

$$g(x) = f(x + 1) - f(x), \quad \text{for } x \in [0, n - 1].$$

If $g(0) = 0$, then $f(1) = f(0)$ so that $x_2 = 1$ and $x_2 = 0$ and we are done. So let us suppose $g(0) > 0$. Then $f(1) > f(0)$. If additionally $f(k+1) > f(k)$ for $k = 1, 2, \ldots, n-1$, then we would get

$$f(0) < f(1) < f(2) \ldots < f(n) = f(0)$$

which is a contradiction. It therefore follows that there is a $k_0 \in \{0, 1, 2, \ldots, n-1\}$ such that $g(k_0) > 0$ but $g(k_0+1) \leq 0$. Since g is continuous, there is $x_0 \in (k_0, k_0+1]$ for which $g(x_0) = 0$. Consequently, $f(x_0 + 1) = f(x_0)$. Set $x_1 = x_0$ and $x_2 = x_0 + 1$. Analogous reasoning can be applied when $g(0) < 0$. □

24. For $n \in \mathbb{N}$, let f be continuous on $[0, n]$ such that $f(0) = f(n)$. Prove that the equation $f(x) = f(y)$ has at least n solutions with $x - y \in \mathbb{N}$.

Solution: Without loss of generality we can assume that $f(0) = f(n) = 0$. The case $n = 1$ is obvious. So suppose that $n > 1$. Consider first the case where $f(1) > 0$, $f(2) > 0$, ..., $f(n-1) > 0$. For $k = 1, 2, 3, \ldots, n-1$, we set $g_k(x) = f(x+k) - f(x)$. The function $g_k(x)$ is continuous on $[0, n-k]$, and by assumption $g_k(0) > 0$ and $g_k(n-k) < 0$. Consequently, there is $x_k \in [0, n-k]$ such that $g_k(x_k) = 0$, or in other words, $f(x_k + k) = f(x_k)$. This shows that the assertion is true in this case. In a similar way, it can be shown to be true in case $f(1) < 0$, $f(2) < 0$, ..., $f(n-1) < 0$.

Suppose now that $f(1) > 0$ and the numbers $f(1)$, $f(2)$, ..., $f(n-1)$ are distinct and non-zero, and there is m, $2 \le m \le n-1$ with $f(m) < 0$. Then there are integers $k_1, k_2, \ldots, k_s$ between 1 and $n-2$ such that

$$\begin{aligned}
&f(1) > 0, f(2) > 0, \ldots, f(k_1) > 0,\\
&f(k_1+1) < 0, f(k_1+2) < 0, \ldots, f(k_2) < 0,\\
&\vdots\\
&f(k_s+1) < 0, f(k_s+2) < 0, \ldots, f(n-2) < 0.
\end{aligned}$$

Now reasoning as in the previous paragraph, we get k_1 solutions in $[0, k_1+1]$, $k_2 - k_1$ solutions in $[k_1, k_2+1]$, and so on. Clearly, all these solutions must be distinct and therefore the assertion is true. If $f(1) < 0$ and $f(m) > 0$ can be similarly dealt.

Finally, consider the case where there are integers k and m, $0 \le k < m \le n$, with $f(k) = f(m)$. Suppose also that the numbers $f(k)$, $f(k+1)$, ..., $f(m-1)$ are distinct. It follows from the above that there are $m-k$ solutions in $[k, m]$. Next define

$$f_1(x) = \begin{cases} f(x) & \text{if } 0 \le x \le k,\\ f(x+m-k) & \text{if } k < x \le n-(m-k). \end{cases}$$

Clearly, f_1 is continuous on $[0, n-(m-k)]$ and $f_1(n-(m-k)) = f(n) = f(0) = f_1(0)$. If $f_1(0)$, $f_1(1)$, ..., $f_1(n-(m-k)-1)$ are distinct, then there are $n-(m-k)$ solutions so that together with the $m-k$ solutions give us n distinct solutions. If some of the numbers $f_1(0)$, $f_1(1)$, ..., $f_1(n-(m-k)-1)$ coincide, the procedure can be repeated. □

25. Suppose that real continuous functions f and g defined on $\mathbb{R}$ commute; that is $f(g(x)) = g(f(x))$ for $x \in \mathbb{R}$. Prove that if the equation $f(f(x)) = g(g(x))$ has a solution, then the equation $f(x) = g(x)$ also has.

Show by example that the assumption of continuity of f and g cannot be omitted.

Solution: Suppose, on the contrary, the equation $f(x) = g(x)$ has no solution. Then $h(x) = f(x) - g(x)$ would be either strictly positive or negative. So that

$$\begin{aligned}
0 &\ne h(f(x)) + h(g(x))\\
&= f(f(x)) - g(f(x)) + f(g(x)) - g(g(x))\\
&= f(f(x)) - g(g(x)),
\end{aligned}$$

which is a contradiction.

The following equation shows that the assumption of continuity is essential:

$$f(x) = \begin{cases} \sqrt{2} & \text{if } x \text{ is irrational},\\ 0 & \text{if } x \text{ is rational}. \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational},\\ \sqrt{2} & \text{if } x \text{ is rational}. \end{cases}$$

□

26. Prove that a continuous injective function $f : \mathbb{R} \to \mathbb{R}$ is either strictly decreasing or strictly increasing.

Solution: Assume, on the contrary, that there are x_1, x_2, and x_3 such that $x_1 < x_2 < x_3$ and, for example, $f(x_1) > f(x_2)$ while $f(x_2) < f(x_3)$. By the intermediate value property, for every u such that $f(x_2) < u < \min\{f(x_1), f(x_3)\}$, there are $s \in (x_1, x_2)$ and $t \in (x_2, x_3)$ satisfying $f(s) = u = f(t)$. Since f is injective, $s = t$, contrary to the fact that $x_1 < s < x_2 < t < x_3$. □

27. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous injective function. Prove that if there exists n such that the n-th composition of f is an identity, that is, $f^n(x) = x$ for all $x \in \mathbb{R}$, then

(a) $f(x) = x$, $x \in \mathbb{R}$, if f is strictly increasing,

(b) $f^2(x) = x$, $x \in \mathbb{R}$, if f is strictly decreasing.

The n-th composition is recursively defined as $f^1(x) = f(x)$ and for each natural $n \ge 2$, $f^n(x) = f(f^{n-1}(x))$.

Solution: From the result of the previous problem, we conclude that f is either strictly increasing or strictly decreasing.

(a) Suppose that f is strictly increasing and there is x_0 such that $f(x_0) \ne x_0$. If $f(x_0) > x_0$, then $f^n(x_0) > x_0$, a contradiction. Similarly, if $f(x_0) < x_0$, then $f^n(x_0) < x_0$, again a contradiction as its given that $f^n(x) = x$ for all $x \in \mathbb{R}$.

(b) If f is strictly decreasing, then $f^2(x)$ is strictly increasing. Since $f^n(x) = x$, we get $f^{2n}(x) = x$, which means that the n-th composition of f^2 is the identity. Therefore, by (a), $f^2(x) = x$.

□

28. Assume that $f :\to \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(f(x)) = -x$, $x \in \mathbb{R}$. Show that f cannot be continuous.

Solution: Notice that f is injective because, if $f(x_1) = f(x_2)$, then we get $-x_1 = f(f(x_1)) = f(f(x_2)) = -x_2$, so that $x_1 = x_2$. It follows from Problem 26 that if f were continuous, then it would be either strictly increasing or strictly decreasing. In both cases, $f(f(x))$ would be strictly increasing which is a contradiction to the fact that $f(f(x)) = -x$ which is strictly decreasing. □

29. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which have the intermediate value property and there exists $n \in \mathbb{N}$ for which the n-th composition $f^n(x) = -x$, $x \in \mathbb{R}$.

Solution: As in the solution of the foregoing problem, it can be shown that f is injective on $\mathbb{R}$. Analysis similar to that in the solution of Problem 26 shows that f is either strictly increasing or strictly decreasing. In both cases f^{2k}, where $k \in \mathbb{N}$, is strictly increasing. Consequently, the integer n in the condition $f^n(x) = -x$ must be odd. If f were strictly increasing, f^n would also be strictly increasing, which would contradict our condition. So f is strictly decreasing. Moreover, since

$$f(-x) = f(f^n(x)) = f^n(f(x)) = -f(x),$$

we see that f is an odd function.

We next show that $f(x) = -x$, $x \in \mathbb{R}$. Suppose that there is an x_0 such that $x_1 = f(x_0) > -x_0$, or in other words, $-x_1 < x_0$. It then follows that $x_2 = f(x_1) < f(-x_0) = -f(x_0) = -x_1 < x_0$. By induction, it can be shown that if $x_k = f(x_{k-1})$, then $(-1)^n x_n < x_0$, which contradicts our assumption that $x_n = f^n(x_0) = -x_0$. Similar reasoning applies to the case where $f(x_0) < -x_0$. Hence $f(x) = -x$ for all $x \in \mathbb{R}$. □

30. A continuous $f : [0,1] \to \mathbb{R}$ attains each of its values infinitely many times and $f(0) \neq f(1)$. Show that f attains at least one of its values an odd number of times.

Solution: We first show that there are at most countably many strict local extrema of f. Indeed, if $x_0 \in (0,1)$ and $f(x_0)$ is a strict local maximum of f, then there exists an interval $(p,q) \subset [0,1]$ with rational endpoints such that $f(x) < f(x_0)$ for $x \neq x_0$ and $x \in (p,q)$. (Similarly for a local minimum). Consequently, our assertion follows from the fact that there are countably many intervals with rational endpoints.

Since there are at most countably many strict local extrema of f, there is a y between $f(0)$ and $f(1)$ which is not an extremal value of f. Assume that $f(0) < f(1)$ and let $y = f(x_1) = f(x_2) = \ldots = f(x_n)$ where $x_1 < x_2 < \ldots < x_n$. Moreover, set $x_0 = 0$ and $x_{n+1} = 1$. Then the function $g(x) = f(x) - y$ is either positive or negative on each interval (x_i, x_{i+1}), and signs are different in adjacent intervals. Notice that the function g is negative in the first interval and positive in the last one. Therefore the number of these intervals is even. Consequently, n is odd. □