Miscellaneous Problems

1. Given the continuous periodic function $f(x)$, $x \in \mathbb{R}$, can we assert that the antiderivative of $f(x)$ is a periodic function?

2. The function $f(x)$ is defined and continuous on the interval $[-a, a]$ with $f(x) = -f(-x)$ for every $x \in [-a, a]$. Prove that every antiderivative $F(x)$ of the function $f(x)$ is an even function.

3. Under what condition(s) does the value of the integral $\int_a^b f(x)\,dx$ coincide with the value of the area of the curvilinear trapezoid bounded by the curves $y = f(x)$, $x = a$, $x = b$ and $y = 0$?

4. The function $y = 1 + \cos x$ and $y = 1 + \cos(x - \alpha)$ where $0 < \alpha < \pi/2$, are given on the interval $[0, \pi]$. For what value of α is the figure bounded by the curves $y = 1 + \cos x$, $y = 1 + \cos(x - \alpha)$ and $x = 0$, equivalent in area to the figure bounded by the curves $y = 1 + \cos(\alpha - x)$, $y = 1$, and $x = \pi$?

5. At what values of the parameter $a > 0$ is the area of the figure bounded by curves $x = a$, $y = 2^x$ and $y = 4^x$ larger or equal to the area of the figure bounded by the curves $y = 2^x$, $y = 0$, $x = 0$ and $x = a$?

6. The *critical* points of a function are the points in the domain of the function at which the derivative of the function vanishes or does not exist. Find the critical points of the function $f(x)$ if

(a) $f(x) = 1 + x + \int_1^x (\ln^2 t + 2\ln t)\,dt$

(b) $f(x) = x - \ln x + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right)\,dx$

7. On the interval $[5\pi/4, 4\pi/3]$, find the least value of the function

$$F(x) = \int_{5\pi/4}^x (3\sin t + 4\cos t)\,dt$$

8. For the function $f(x) = 1 + 3^x \ln 3$, find the antiderivative $F(x)$ which assumes the value 7 for $x = 2$. At what values of x does the curve $y = F(x)$ cut the abscissa axis?

9. Prove the identity $\cos 2x = \cos^2 x - \sin^2 x$ by differentiating, term by term, the identity $\sin 2x = 2\sin x \cos x$.

10. At what positive values of a do the graphs of the function $f(x) = a|x + 1|$ and $g(x) = x + a^2|x|$ meet at three distinct points?

11. Find all the values of φ for which the sum of the squares of the roots of the equation $x^2 + (\sin\varphi - 1)x - \frac{1}{2}\cos^2\varphi = 0$, is greatest.

12. At what values of the parameter a belonging to the interval $[5\pi/6, \pi]$ does the quadratic trinomial $(\cot a)x^2 + 2x\sqrt{\tan a} + \tan a$ assume only positive values?

13. Find all the values of the parameter α for which the quadratic function $(\sin\alpha)x^2 + 2x\cos\alpha + (\cos\alpha + \sin\alpha)/2$ is the square of a linear function.

14. Find all values of b for which the equation

$$2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$$

has only one solution.

15. At what values of a is any solution of the inequality

$$\frac{\log_3(x^2 - 3x + 7)}{\log_3(3x + 2)} < 1$$

also a solution of the inequality $x^2 + (5 - 2a)x \le 10a$?

16. Are the inequalities $f(x) > g(x)$ and $f'(x) > g'(x)$ equivalent?

17. Find the numbers a, b, c such that the function of the form $f(x) = ax^2 + bx + c$ satisfies the conditions $f'(1) = 8$, $f(2) + f''(2) = 33$, and $\int_0^1 f(x)\,dx = \frac{7}{3}$.

18. Find the area of the figure bounded by the parabola $y = -x^2 + 7x - 12$, the tangent to this parabola through its vertex and the coordinate axes.

19. Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$, if the tangent to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1} 6$ with the x axis.

20. Find the area of the figure bounded by the curve $y = 9^{-x} + 85$ and the curve $y = k\,3^{-x} + m$ passing through the points $(0, 34)$ and $(1, 14)$.

21. Generally, it is convenient to investigate a function and construct its graph according to the following steps:

- Find the domain of the function.

- Find out whether the function is even, odd or periodic.
- Test the function for continuity; find out the discontinuities and their character (whether they are finite or infinite discontinuities; whether removable or jump).
- Find the asymptotes of the function i.e. its behavior as $x \to \pm\infty$ or near the points of discontinuities and non–differentiability.
- Find the critical points and calculate the maximum and minimum values of the function (if any); find whether the function increases or decreases between these critical points.
- Plot the approximate graph trying to be as accurate as you can.

The steps sketched above are not, by any means, compulsory; and of course many alternative routes can be taken. Using the above general plan, sketch the graph of the following functions:

(a) $y = x^6 - 3x^4 + 3x^2 - 5$ (b) $y = \sqrt[3]{x} - \sqrt[3]{x+1}$
(c) $y = \dfrac{2x^3}{x^2 - 4}$ (d) $y = \dfrac{1 - x^3}{x^2}$
(e) $y = x + \ln(x^2 - 1)$ (f) $y = \frac{1}{2}\sin 2x + \cos x$
(g) $y = x^2 e^{1/x}$ (h) $y = \sin^{-1}\dfrac{1 - x^2}{1 + x^2}$

22. Suppose that f is a continuous real valued function. Show that for some $\xi \in [0,\, 1]$

$$\int_0^1 f(x)x^2\,\mathrm{d}x = \frac{1}{3}\,f(\xi)$$

23. Let $f(x) = x\ln(1 + x^{-1})$, $0 < x < \infty$. Show that f is strictly monotonically increasing; and calculate $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

24. Let $f(x)$, $0 \le x < \infty$, be continuous and differentiable and suppose that $f(0) = 0$ and that $f'(x)$ is an increasing function of x for $x \ge 0$. Prove that

$$g(x) = \begin{cases} f(x)/x & \text{if } x > 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

is an increasing function of x. Interpret the result pictorially.

25. Let k be a real constant. Suppose that $y(t)$ is a positive differentiable function satisfying $y'(t) \le ky(t)$ for $t \ge 0$. Prove that $y(t) \le e^{kt}y(0)$ for $t \ge 0$.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and suppose that for all $x \in \mathbb{R}$, $|f(x)| \le 1$ and $|f''(x)| \le 1$. Prove that $|f'(x)| \le 2$ for all $x \in \mathbb{R}$.

27. Let f be a positive differentiable function on $(0,\, \infty)$. Prove that

$$\lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h}$$

exists (finitely) and is nonzero for each x.

28. Suppose $f : [0,\, 1] \to \mathbb{R}$ is continuous with $f(0) = 0$, and for $0 < x < 1$, f is differentiable and $0 \le f'(x) \le 2f(x)$. Prove that f is identically 0.

29. Let $f : [0,\, \infty) \to [0,\, \infty)$ be a continuous, strictly increasing function and let $g = f^{-1}$. Prove that

$$\int_0^a f(x)\,\mathrm{d}x + \int_0^b g(y)\,\mathrm{d}y \ge ab$$

30. Show that if $f(a) = f(b) = 0$ and f is twice differentiable, then

$$\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x$$

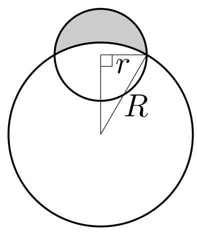
31. Evaluate the following integrals:

(a) $\displaystyle\int_0^2 x^3\sqrt{4 - x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 + 4x + 8}}$

(c) $\displaystyle\int \frac{\mathrm{d}x}{(5 - 4x - x^2)^{5/2}}$

32. Find the area of the crescent shaped region bounded by arcs of circles with radii r and R, as in the figure below.



33. Find the area of the region bounded by the curves $x^2 + (y-1)^2 = 2$ and $y = \frac{1}{2}(x^2 - 1)$.

34. Find the equation of the line which is tangent to the curve

$$y = -x^4 + 18x^3 - 97x^2 + 180x - 52$$

at exactly two distinct points. Also find the points of tangency.

35. Evaluate:

$$\lim_{n\to\infty}\left(\int_1^3 x^{n/x}\mathrm{d}x\right)^{1/n}$$

36. If f be a continuous function defined on $\mathbb{R}$ and satisfying $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$, determine the following limit:

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{f(x)}\,\mathrm{d}x\right)^n$$

37. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there exists a real number $a > 0$ such that $f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$ for all $x > 0$.

38. If $P_1(x) = 1 + x$ and

$$P_n(x) = P_{n-1}(x)(1 + P'_{n-1}(x))$$

for $n = 2, 3, 4, \ldots$, then find $P_n(0)$ and $P'_n(0)$.

39. If $F_1(x) = 1 + x$ and

$$F_n(x) = F_{n-1}(x)\left(1 + \int_0^x F_{n-1}(t)\mathrm{d}t\right)$$

for $n = 2, 3, 4, \ldots$, then find $F_n(0)$ and $F'_n(0)$.

40. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$, $f(0) = 0$ and $0 \le f'(x) \le 1$ for all $x \in (0, 1)$. Prove that

$$\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \ge \int_0^1 (f(x))^3\mathrm{d}x$$

41. Let $|\cos\theta| \ne |\sin\theta|$. Find the value of k satisfying the following equation:

$$\int_{\cos^2\theta}^{\sin^2\theta}(x - \cos^2\theta)(x - \sin^2\theta)(x - k)\mathrm{d}x = 0$$

42. Let f be a continuously differentiable function, such that $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$. Prove that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(f(x))^2\mathrm{d}x \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(f'(x))^2\mathrm{d}x$$

43. For a real number λ, find the solutions (i.e. find the functions $f(x)$ and $g(x)$) satisfying

$$f(x) = e^x + \lambda\int_0^x e^{x-y}\, f(y)\mathrm{d}y, \quad 0 \le x \le 1$$
$$g(x) = e^x + \lambda\int_0^1 e^{x-y}\, g(y)\mathrm{d}y, \quad 0 \le x \le 1$$

44. Let $f, g : [0, 1] \to [0, 1]$ be continuous. Assume that f is non-decreasing. Prove that

$$\int_0^1 f(g(x))\mathrm{d}x \le \int_0^1 f(x)\mathrm{d}x + \int_0^1 g(x)\mathrm{d}x$$

45. Let $a > 0$ be a real number and $f : [-a, a] \to \mathbb{R}$ such that f'' exists and is integrable. Suppose $f(a) = f(-a)$, and $f'(-a) = f'(a) = a^2$. Prove that

$$\int_{-a}^{a}(f''(x))^2\,\mathrm{d}x \ge 6a^3$$

When does the equality hold?