Assignment M–11–4

(Permutations and combinations)

1. Out of 7 boys and 4 girls, a committee of 6 is to be formed; in how many ways can this be done, (i) when the committee contains exactly 2 girls, (ii) at least 2 girls?

Ans: (i) ${}^4C_2 \times {}^7C_4$, (ii) ${}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2$ ☐

2. Out of 7 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels? The words need not have any meaning. (*Ans:* ${}^7C_3\,{}^4C_2\ 5!$)

3. In how many ways can you take an odd number of objects from n objects?

Ans: ${}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$ ☐

4. How many ways can you split 14 people into 7 pairs?

Ans: $\dfrac{1}{7!}\,{}^{14}C_2\,{}^{12}C_2\,{}^{10}C_2 \cdots {}^2C_2 = \dfrac{14!}{7!(2!)^7}$ ☐

5. There are n boys and n girls in a dance class. How many ways are there to pair them all up? (*Ans:* $n!$)

6. One student has 6 books and another has 8. In how many ways can they exchange 3 books of the first student for 3 books of the second? (*Ans:* ${}^6C_3\,{}^8C_3$)

7. A group of soldiers contains 3 officers, 6 sergeants, and 30 privates. How many ways can a team be formed consisting of 1 officer, 2 sergeants, and 20 privates? (*Ans:* ${}^3C_1\,{}^6C_2\,{}^{30}C_{20}$)

8. A person has 10 friends. Over several days he invites some of them to a dinner party in such a way that he never invites exactly the same group of people. How many days can he keep this up, assuming that one of the possibilities is to ask nobody to dinner?

Ans: Same as the number of subsets (including the null set) of a set containing 10 objects which is $2^{10} = 1024$. ☐

9. Each of the faces of a cube is coloured by a different color. How many of the colourings are distinct?

Sol: Let the colors (and the faces on which they are put) be called 1, 2, 3, 4, 5, and 6. Put the cube on the table so that face 1 is at the bottom. Consider color 2. There are only two possibilities: either 2 is opposite 1 or it is adjacent to 1. If 2 is opposite 1, we can rotate the cube about a vertical axis so that face 3 is in front. Now the cube is fixed. There are $3! = 6$ ways to complete the colouring. If face 2 is adjacent to face 1, rotate the cube so that face 2 is in the front. Now the cube is fixed and the colouring can be completed in $4! = 24$ ways. Altogether, there are $6 + 24 = 30$ distinct colourings of the cube by using six different colors. ☐

10. In how many ways can n things be given to p persons, when there is no restriction as to the number of things he may receive? (*Ans:* p^n)

11. In how many ways can 7 men and 7 women sit down at a round table, no two women together? (*Ans:* 6! 7!)

12. Find the number of ways of dividing mn things into n equal groups.

Ans: $\dfrac{(mn)!}{n!(m!)^n}$ ☐

13. Find the number of permutations which can be formed out of the letters of the word *series* taken three together.

Ans: 42. (Count each of the following cases: (i) all three letters different, and (ii) two are identical and one different.) ☐

14. In how many ways can 5 horses finish a race.

Ans: 541. Let H_n denote the number of ways in which n horses can finish the race and define $H_0 = 1$. Then we have the following recurrence: $H_n = \sum_{k=1}^{n} {}^nC_k H_{n-k}$. The first few numbers in the sequence of H_n are 1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853, 5315654681981355, 130370767029135901, 3385534663256845323, 92801587319328411133, 2677687796244384203115, . . . ☐

15. There are n different books and p copies of each. Find the number of ways in which a selection can be made from them. (*Ans:* $(p+1)^n - 1$)

16. Find the number of selections and arrangements that can be made by taking 4 letters from the word *expression*.

Ans: Selections = 113, permutations = 2190 □

17. Let R_n be the number of ways of placing n identical rooks on an $n \times n$ chessboard so that they do not attack each other. Find a formula for R_n. Further, let M_n and D_n be the number of those placing which remain unchanged upon reflections about a diagonal and about both diagonals respectively. Prove that $M_n = M_{n-1} + (n-1)M_{n-2}$, and $D_{2n} = 2D_{2n-2} + (2n-2)D_{2n-4}$, $D_{2n+1} = D_{2n}$ for integers n. (*Ans:* $R_n = n!$)

18. If of $p + q + r$ things, p be alike of one kind, q be alike of different kind and rest all be different, find the total number of selections that can be made. (*Ans:* $(p+1)(q+1)2^r - 1$)

19. $2n$ objects of each of three kinds are given to two persons, so that each person gets $3n$ objects. Prove that this can be done in $3n^2 + 3n + 1$ ways.

20. If the $n+1$ numbers a_0, a_1, a_2, a_3, $\ldots$, a_n be all different, and each of them a prime number, find the number of different factors of the expression $a_0^m a_1 a_2 a_3 \cdots a_n$. (*Ans:* $(m+1)2^n$)

21. Of $3n+1$ objects, n are indistinguishable, and the remaining ones are distinct. Show that one can choose from them n objects in 2^{2n} ways.

22. Find the sum $S_n = \sum_{k=1}^{n} {}^nC_k\, k^3$. Hint: the sum can be interpreted as the number of ways to choose a committee of at least one that includes a president, vice-president, and treasurer (not necessarily distinct persons) from a set of n people.

Ans: $S_n = n^2(n+3)2^{n-3}$ □

23. Ten places in a row are each to be filled either with a 0, a 1, or a 2. How many ways can this be done if no two or more neighbouring 0's should appear?

Ans: 24960. If a_n denotes the number of such placements in n places, then the following recurrence is valid: $a_n = 2(a_{n-1} + a_{n-2})$. Defining $a_0 \equiv 1$ (and easily counting) $a_1 = 3$, the first few terms in the sequence are: 1, 3, 8, 22, 60, 164, 448, 1224, 3344, 9136, 24960, 68192, 186304, 508992, 1390592, 3799168, 10379520, 28357376, 77473792, 211662336, 578272256, 1579869184, 4316282880, 11792304128, 32217174016, 88018956288, 240472260608, 656982433792, 1794909388800, 4903783645184, 13397386067968, ... The formula for a_n can be obtained in a closed form: $a_n = \frac{1}{4\sqrt{3}}\left((1+\sqrt{3})^{n+2} - (1-\sqrt{3})^{n+2}\right)$, for $n = 0, 1, 2 \ldots$. □

24. Find the highest power of 6 that is contained in 100!. (*Ans:* 48)

25. Six papers are set in an examination, two of them in mathematics. In how many different orders can the papers be given if the two mathematics paper are not successive? (*Ans:* $6! - 2(5!) = 480$)

26. We draw all diagonals of a convex n-sided polygon. Suppose no three diagonals pass through a point. Into how many regions is the interior of the polygon divided? (*Ans:* $1 + {}^n\mathrm{C}_2 - n + {}^n\mathrm{C}_4$)

27. It is a rule in Gaelic that no consonant or group of consonants can stand immediately between a strong and a weak vowel; the strong vowel being *a, o, u* and the weak vowels *e* and *i*. Show that the total number of Gaelic words of $n+3$ letters each, each of which can be formed of n consonants and the vowels *a, e, o*, where no letter is repeated in the same word is $2\frac{(n+3)!}{n+2}$.

28. $2n$ points are given on a circle. In how many ways can you join pairs of points by non intersecting chords?

Ans: $\frac{{}^{2n}\mathrm{C}_n}{n+1}$ □

29. We have 20 presents that we want to distribute to 12 children. It is not necessary that every child gets something; in fact we might end up giving all the presents to a single child. In how many ways can we distribute the presents? (*Ans:* 12^{20})

30. Suppose we have n different presents, which we want to distribute to k children. Let's call the children as 1, 2, 3, $\ldots k$ (not a very friendly way to name them!). We want that child 1 should get exactly n_1 presents, child 2 should get exactly n_2 presents, and so on and finally the last child gets n_k presents. Obviously, we must have $n_1 + n_2 + n_3 + \ldots + n_k = n$. How many ways can we distribute the presents?

Ans: $\frac{n!}{n_1!n_2!\cdots n_k!}$ □