Counting Techniques

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Contents

1 Introduction

We begin by studying two very simple examples.

***Example* 1.** Consider the collection of all 2–digit numbers where the first digit is either 1 or 2, and where the second digit is either 6, 7 or 8. Clearly there are 6 such numbers and they are listed below:

16	17	18
26	27	28

Arranged this way, we note that each row corresponds to a choice for the first digit and each column corresponds to a choice for the second digit. We have 2 rows and 3 columns, and hence $2 \times 3 = 6$ possibilities.

***Example* 2.** Consider the collection of all 3–digit numbers where the first digit is either 1, 2, 3 or 4, where the second digit is either 5 or 6, and where the third digit is either 7, 8 or 9. The candidates are listed below:

157	158	159	257	258	259	357	358	359	457	458	459
167	178	179	267	268	269	367	368	369	467	468	469

Arranged this way, we note that each block corresponds to a choice for the first digit. Within each block, each row corresponds to a choice for the second digit and each column corresponds to a choice for the third digit. We have 4 blocks, each with 2 rows and 3 columns, and hence $4 \times 2 \times 3 = 24$ possibilities.

These two examples are instances of a simple but very useful principle.

FUNDAMENTAL PRINCIPLE OF COUNTING. Suppose that an operation can be performed in m different ways, and once it has been performed, another operation can be performed in n different ways, then the number of different ways for these two operations can be performed in succession is $m \cdot n$.

The above principle can be extended to any number of successive operations.

***Example* 3.** Consider motor vehicle licence plates made up of 3 letters followed by 3 digits, such as $ABC012$. To determine the total number of possible different licence plates, note that there are 26 choices for each letter and 10 choices for each digit. Hence the total number is $26\times26\times26\times10\times10\times10$. On the other hand, if the first digit is restricted to be non–zero, then the total number is only $26 \times 26 \times 26 \times 9 \times 10 \times 10$. Furthermore, if the letters are required to be dissimilar and the first digit is restricted to be non–zero, then the total number is only $26 \times 25 \times 24 \times 9 \times 10 \times 10$. Finally, if the letters and digits are required to be dissimilar and the first digit is restricted to be non–zero, then the total number is only $26\times25\times24\times9\times9\times8$.

2 Permutations & Combinations

▷ **Definition 1 (Permutation).** Each of the *arrangements* which can be made by taking some or all of a given number of dissimilar objects is called a *permutation.*

▷ **Definition 2 (Combination).** Each of the *selections* or *groups* which can be made by taking some or all of a given number of dissimilar objects is called a *combination.*

For example, all possible permutations of the letters a, b and c when we take them all together are as follows:

$$\begin{matrix} abc & bac & cab \\ acb & bca & cba \end{matrix}$$

Hence, there are 6 different permutations. But if we talk of groups or selections, then there is only one: abc because in any other arrangement we have the same letters and hence no different selections. To make permutations of only two letters out of a, b and c, they are

$$\begin{matrix} ab & ba & ca \\ ac & bc & cb \end{matrix}$$

In this case however, we can make three selections: ab, ac and bc.

From the above examples, it seems that while considering the number of permutations, *ordering is important* while for combinations the ordering does not matter. Generalizing this observation we can represent the permutation of r dissimilar objects taken from a total of n dissimilar objects as ordered $r-$ tuple while the combination of r dissimilar objects taken from a total of n dissimilar objects can be represented as a set containing r elements.

2.1 Basic Results

↬▶ **1. *To find the number of permutations of n dissimilar objects taken r at a time.***

This is the same as finding out the number of ways in which we can fill up r places when we have n dissimilar objects at our disposal.
The first place can be filled up in n ways, for we can take any of the n objects. When it has been filled up in any one of these ways, the second place can then be filled up in $n-1$ ways; and since each way of filling up the first place can be associated with each way of filling the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$. And when the first two places have been filled up in any way, the third place can be filled up in $n-2$ ways. And reasoning as before, the number of ways in which three places can be filled up is $n(n-1)(n-2)$.
Proceeding thus, and noting that a new factor is introduced with each new place filled up, and that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to

$$n(n-1)(n-2)\cdots\cdots \text{to } r \text{ factors};$$

and the r–th factor is

$$n-(r-1)=n-r+1.$$

Therefore, the number of permutations of n dissimilar objects taken r at a time is

$$n(n-1)(n-2)\cdots\cdots(n-r+1). \tag{1}$$

◇ **Corollary 1.** The number of permutations of n dissimilar objects taken all at a time is

$$n(n-1)(n-2)\cdots\cdots \text{to } n \text{ factors} = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1.$$

It is usual to denote this product by the symbol $n!$ (read: "n factorial") which is the product of natural numbers from 1 to n.

♣ *Remark* 1. From now on, the number of permutations of n things taken r at a time will be denoted by symbol ${}^n\mathrm{P}_r$ (read: "n p r"), so that

$$\begin{aligned} {}^n\mathrm{P}_r &= n(n-1)(n-2)\cdots\cdots(n-r+1)\\ &= \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)(n-r)!}{(n-r)!}\\ &= \frac{n!}{(n-r)!}\end{aligned}$$

♣ *Remark* 2. If we want to find out the number of permutations of zero objects taken all (i.e. zero objects) at once, how many ways are there. Well, there is only one way to arrange nothing — do nothing. And hence, we have

$$0!=1, \quad \text{so that} \quad {}^n\mathrm{P}_r=\frac{n!}{0!}=n!.$$

♣ *Remark* 3. The number of permutations of n things taken r at a time can also be found in the following way.
Let ${}^n\mathrm{P}_r$ represent the number of permutations of n things taken r at a time. Suppose, we have formed all permutations of n things taken $r-1$ at a time; the number of these will be ${}^n\mathrm{P}_{r-1}$.
With *each of these* put one of the remaining $n-r+1$ things. Each time we do so, we shall get one permutation of n things r at a time; and therefore the whole number of the permutations of n things r at a time is ${}^n\mathrm{P}_{r-1} \times (n-r+1)$; that is

$$ {}^n\mathrm{P}_r = {}^n\mathrm{P}_{r-1} \times (n-r+1). $$

By writing $r-1$ for r in this formula, we obtain

$$ \begin{aligned} {}^n\mathrm{P}_{r-1} &= {}^n\mathrm{P}_{r-2} \times (n-r+2), \\ \text{similarly} \qquad {}^n\mathrm{P}_{r-2} &= {}^n\mathrm{P}_{r-3} \times (n-r+3), \\ \cdots\cdots & \quad \cdots\cdots \\ {}^n\mathrm{P}_3 &= {}^n\mathrm{P}_2 \times (n-2), \\ {}^n\mathrm{P}_2 &= {}^n\mathrm{P}_1 \times (n-1), \\ {}^n\mathrm{P}_1 &= n. \end{aligned} $$

Multiplying together the vertical columns and canceling the like factors from each side, we obtain

$$ {}^n\mathrm{P}_r = n(n-1)(n-2)\cdots\cdots(n-r+1). $$

➥ 2. *To find out the number of combinations of n dissimilar objects taken r at a time.*

The number of combinations of n dissimilar objects taken r at a time is denoted by ${}^n\mathrm{C}_r$ (read: "n c r"). Each of these combinations consists of a group of r dissimilar objects which can be arranged among themselves in $r!$ ways.
Hence, ${}^n\mathrm{C}_r \times r!$ is equal to the number of *arrangements* of n things taken r at a time; that is

$$ \begin{aligned} {}^n\mathrm{C}_r \times r! &= {}^n\mathrm{P}_r = n(n-1)(n-2)\cdots\cdots(n-r+1), \\ \therefore \quad {}^n\mathrm{C}_r &= \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!} \end{aligned} \tag{2} $$

If we multiply the numerator and denominator by $(n-r)!$, we obtain

$$ {}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}\,. \tag{3} $$

***Example* 4.** From 12 books, in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?
Solution: (1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11. Hence the number of ways

$$ = {}^{11}\mathrm{C}_4 = \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} = 330. $$

(2) Since the specified book is to be always excluded, we have to select 5 out of remaining 11 books. Hence the number of ways

$$ = {}^{11}\mathrm{C}_5 = \frac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5} = 462. $$

➥ 3. *The number of combinations of n things r at a time is equal to the number of combinations of n things $n-r$ at a time.*

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n-r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n-r$ at a time:

$$ {}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}. \tag{4}$$

Alternatively, we have

$$\begin{aligned} {}^n\mathrm{C}_{n-r} &= \frac{n!}{(n-r)!\,(n-(n-r))!} \\ &= \frac{n!}{(n-r)!\,r!} \\ &= {}^n\mathrm{C}_r \end{aligned}$$

Such combinations are called *complementary.*

♣ *Remark* 4. Put $r = n$, then ${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n = 1$.

↬▶ **4. *To find the number of way in which* $m+n$ *different objects can be divided into two groups containing* m *and* n *objects severally.***

This is clearly equivalent to finding the number of combinations of $m+n$ things taken m at a time, because every time we select one group of m objects we leave a group of n things behind. Thus the required number

$$= \frac{(m+n)!}{m!\; n!}\,. \tag{5}$$

♣ *Remark* 5. If $n = m$, the groups are equal, and in this case the number of *different* ways of subdivision is $\dfrac{(2m)!}{2!\; m!\; m!}$; for in any way it is possible to interchange the two groups without obtaining a new distribution.

↬▶ **5. *To find the number of ways in which* $m+n+p$ *different objects can be divided into three groups containing* m*,* n*, and* p *objects severally.***

First divided $m+n+p$ objects into two groups containing m and $n+p$ objects respectively. The number of ways in which this can be done is

$$\frac{(m+n+p)!}{m!\;(n+p)!}\,.$$

Then the number of ways in which the group of $n+p$ objects can be divided into two groups containing n and p objects respectively is

$$\frac{(n+p)!}{n!\; p!}\,.$$

Hence, the number of ways of subdivision into three groups containing m, n and p objects severally is

$$\frac{(m+n+p)!}{m!\;(n+p)!} \times \frac{(n+p)!}{n!\; p!} = \frac{(m+n+p)!}{m!\; n!\; p!}\,.$$

♣ *Remark* 6. If we put $n = p = m$, we obtain $\dfrac{(3m)!}{m!\; m!\; m!}$; but this formula regards as different all the possible orders in which the three groups can occur in any mode of subdivision. And since there are 3! such orders corresponding to any one mode of subdivision, the number of *different* ways in which subdivision into three *equal* groups can be made is $\dfrac{(3m)!}{3!\; m!\; m!\; m!}$.

♣ *Remark* 7. Generally, it is advisable, the direct formula for *permutations* should not be used until the suitable *selections* have been made.

***Example* 5.** From 7 men and 4 women, a committee of 6 is to be formed. In how many ways can this be done, when the committee contains (i) exactly 2 women, (ii) at least 2 women?
Solution: (i) We have to chose 2 women and 4 men. The number of ways in which the women can be chosen in ${}^4C_2$ and the number of ways in which the men can be chosen is ${}^7C_4$. Each of the first groups can be associated with each of the second; hence the required number of ways

$${}^4C_2 \times {}^7C_4 = \frac{4!}{2!\,2!} \times \frac{7!}{4!\,3!} = 210.$$

(ii) The committee may contain 2, 3 or 4 women. We shall exhaust all suitable combinations by forming all the groups containing 2 women and 4 men; then 3 women and 3 men; and lastly 4 women and 2 men. The *sum* of the three results will give the required number. Hence the answer:

$$\begin{aligned}{}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2 &= \frac{4!}{2!\,2!} \times \frac{7!}{4!\,3!} + \frac{4!}{3!\,1!} \times \frac{7!}{3!\,4!} + \frac{4!}{1!\,0!} \times \frac{7!}{2!\,5!} \\ &= 210 + 140 + 21 \\ &= 371.\end{aligned}$$

So far, the objects have been regarded as *unlike* meaning we can visibly distinguish between them. We now relax this bound and allow objects to be visibly identical or indistinguishable.

↬▶ **6. *To find the number of permutations of n objects, taken all at a time, when p of them are exactly alike of one kind, q of them exactly alike of another kind, r of them exactly alike of the third kind, and the rest all different.***

Let there be n letters; suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike.
Let x be the required number of permutations; then if the p letters a were replaced by unlike letters different from any of the rest, from *any one* of the x permutations, without altering the position of any of the remaining letters, we could form $p!$ new permutations. Hence if this change were made in each of the x permutations, we should obtain $x \times p!$ permutations.
Similarly, if the q letters b were replaced by q unlike letters, the number of permutations would be $x \times p! \times q!$.
In similar manner, by replacing the r letters c by r unlike letters, we should finally obtain $x \times p! \times q! \times r!$.
But the things are now all different, and therefore admit of $n!$ permutations among themselves. Hence

$$\begin{aligned} x \times p! \times q! \times r! &= n! \\ \Rightarrow \quad x &= \frac{n!}{p!\,q!\,r!}, \end{aligned} \tag{6}$$

which is the required number of permutations.
Any case in which the objects are not all different can be treated similarly.

***Example* 6.** How many different permutations can be made out of the letters of the word *assassination* taken all together?
Solution: We have 13 letters out of which 4 are s, 3 are a, 2 are i, and 2 are n. Hence the number of permutations

$$\begin{aligned} &= \frac{13!}{4!\,3!\,2!\,2!} \\ &= 13 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 3 \cdot 5 \\ &= 1001 \times 10800 = 10810800. \end{aligned}$$

↬▶ 7. *To find the number of permutations of n objects r at a time, when each object may be repeated once, twice, ... up to r times in any arrangement.*

Here we have to consider the number of ways in which r places can be filled up when we have n different objects at our disposal, each of them being used as often as we please in any arrangement.
The first place may be filled in n ways, when it has been filled up in any of the ways, the second place can be filled up in n ways, since we are not precluded from using the same thing again. Therefore the number of ways in which the first two places can be filled is $n \times n$ or n^2. The third place can also be filled up in n ways, and therefore the first three places in n^3 ways.
Proceeding in this manner, and noticing that at any stage the index of n is always the same as the number of places filled up, we shall have the number of ways in which the r places can be up is equal to n^r.

Example 7. In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?
Solution: Any one prize can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may also be given to the boy who has already received a prize. Thus two prizes can be given in 4^2 ways, and so on. Hence the 5 prizes can be given away in 4^5, or 1024 ways.

↬▶ 8. *To find the total number of ways in which it is possible to make a selection by taking some or all of n distinct objects.*

Each of the objects can be dealt with in two ways. We either take it in our selection or leave it. Since any of these two ways of dealing with an object can be associated with the two ways of dealing with each of the others, the number of ways of dealing with the n objets is

$$\underbrace{2 \times 2 \times 2 \times \cdots\cdots \times 2}_{\text{total of } n \text{ factors}}.$$

But this includes the case in which all the things are left, therefore rejecting this case, the total number of ways is

$$2^n - 1. \tag{7}$$

↬▶ 9. *To find the number of ways in which a selection can be made by taking some or all out of $p + q + r + \cdots\cdots$ objects out of which p are alike of one kind, q are alike of another kind, r are alike of a third kind; and so on.*

The p objects may be disposed in $(p + 1)$ ways; for we may take 0, 1, 2, ..., p of them. Similarly, the q objects may be disposed in $(q + 1)$ ways; for we may take 0, 1, 2, ..., q of them; and so on. Hence, the number of ways in which all the objects might be disposed of is

$$(p + 1)(q + 1)(r + 1)\cdots\cdots$$

But this includes the case in which none of the objects are taken; therefore, rejecting this case, the total number of ways is

$$(p + 1)(q + 1)(r + 1)\cdots\cdots - 1\,.$$

↬▶ 10. *To find the value of r for which the number of combinations of n different objects taken r at a time is greatest.*

If ${}^n\mathrm{C}_r$ is the greatest then we must have ${}^n\mathrm{C}_{r-1} \le {}^n\mathrm{C}_r \ge {}^n\mathrm{C}_{r+1}$. The left inequality gives

$$1 \le \frac{{}^n\mathrm{C}_r}{{}^n\mathrm{C}_{r-1}} = \frac{n!}{r!\,(n-r)!} \cdot \frac{(r-1)!\,(n-r+1)!}{n!} = \frac{n-r+1}{r} = \frac{n+1}{r} - 1$$

That is

$$r \le \frac{n+1}{2} \tag{8}$$

The right inequality gives

$$1 \geq \frac{{}^n\mathrm{C}_{r+1}}{{}^n\mathrm{C}_r} = \frac{n!}{(r+1)!\ (n-r-1)!} \cdot \frac{(r)!\ (n-r)!}{n!} = \frac{n-r}{r+1}$$

That is

$$r \geq \frac{n-1}{2} \tag{9}$$

From Eq. 8 and Eq. 9, we obtain

$$\frac{n-1}{2} \leq r \leq \frac{n+1}{2} \tag{10}$$

Now there are two possibilities:

1. n is **even:** Put $n = 2m$, where m is an integer. Then from Eq. 10, we get

$$m - \frac{1}{2} \leq r \leq m + \frac{1}{2}$$

   That is, r should be an integer lying between $m - \frac{1}{2}$ and $m + \frac{1}{2}$. But the only such integer is m. So in this case we must choose $r = m = n/2$. Thus, for an even n, ${}^n\mathrm{C}_{\frac{n}{2}}$ is greatest among all the ${}^n\mathrm{C}_r$'s.

2. n is **odd:** Put $n = 2m + 1$, where m is any integer. Then we have from Eq. 10

$$m \leq r \leq m + 1$$

   So we can choose the value of r as either of m or $m+1$, i.e. we can choose r as either $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Thus when n is odd, the greatest among ${}^n\mathrm{C}_r$'s are ${}^n\mathrm{C}_{\frac{n-1}{2}}$ and ${}^n\mathrm{C}_{\frac{n+1}{2}}$ both being equal.

3 PIE: Principle of Inclusion and Exclusion

Suppose that we know the sizes of some sets $A_1, A_2, \ldots, A_n$. How many elements are in the union? If the sets are disjoint, then the size of the union is given by the simple Sum Rule. If the sets are not necessarily disjoint, then the size is given by the more general, but more complicated Inclusion-Exclusion Principle.

3.1 The Sum Rule

Lemma 1. *If A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$.*

Lemma 1 could be proved from the denition of cardinality, using properties of bijections and natural numbers. But again, the formal proof is really only of interest to Logicians. So we'll accept Lemma 1 without proof as an axiom. It generalizes straightforwardly to

⊙ **Theorem 2** (Sum Rule)**.** *If $A_1, A_2, \ldots A_n$ are disjoint sets, then:*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n| \tag{11}$$

The Sum Rule says that the number of elements in a union of disjoint sets is equal to the sum of the sizes of all the sets. The Sum Rule can be proved from Lemma 1 by induction on the number of sets.

3.2 Inclusion-Exclusion Principle (Special Cases)

The Sum Rule gives the number of elements of a union of disjoint sets. The Inclusion-Exclusion Principle gives the number of elements of a union of sets that may have common elements. The Inclusion-Exclusion Principle for n sets is messy to write down, so we'll start with the simple special cases $n = 2$ and $n = 3$.

⊙ **Theorem 3** (Inclusion-Exclusion Principle for 2 sets)**.** *Let A and B be sets, not necessarily disjoint. Then*

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{12}$$

The result is easily accounted for. Items in the union of A and B that are in the intersection of A and B are counted twice in the sum $|A| + |B|$. Therefore, by subtracting $|A \cap B|$, every element is counted once overall.

Next, the theorem for three sets.

⊙ **Theorem 4** (Inclusion-Exclusion Principle for 3 sets)**.** *Let A, B, and C be sets, not necessarily disjoint. Then*

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -|A \cap B| - |A \cap C| - |B \cap C| \\ &\quad +|A \cap B \cap C| \end{aligned} \tag{13}$$

Though this formula contains many terms, the general pattern is easy to remember: add the sizes of individual sets (rst line), subtract intersections of pairs of sets (second line), and add the intersection of all three sets (third line). The truthfulness of the statement can be seen easily. Items contained in just one of the sets A, B, or C are counted once on the rst line. Since these items are not contained in any intersection of sets, they are not subtracted away on the second line or counted again on the third line. In total, these items are counted just once.

Items contained in exactly two of the sets A, B, and C (not in all three) are counted twice on the rst line, subtracted away once on the second line, and not counted again on the third line. Again, in total, these items are counted just once.

Items contained in all three sets A, B, and C are counted three times on the rst line, subtracted away three times on the second line, and added back once on the third line. Since $33 + 1 = 1$, these items are also counted just once overall. The name "Inclusion-Exclusion" comes from the way items are counted, subtracted away, counted again, etc.

***Example* 8.** Many of Alice's 16 friends are athletic – they cycle, jog or swim on a regular basis. In fact we know that 6 of them cycle, 6 of them jog, 6 of them swim, 4 of them cycle and jog, 2 of them cycle and swim, 3 of them jog and swim and 2 of them engage in all three activities. How many of Alice's friends do none of these things on a regular basis?

Solution: Let us first find the number of Alice's friends who are engaged in at least one of the activities. To that end, we put her friends into three sets: $A = \{\text{friends who cycle}\}$, $B = \{\text{friends who jog}\}$, and $C = \{\text{friends who swim}\}$. Obviously, some of her friends may belong to more than one of these sets. According to the given information: $|A| = 6$, $|B| = 6$, $|C| = 6$, $|A \cap B| = 4$, $|A \cap C| = 2$, $|B \cap C| = 3$, and $|A \cap B \cap C| = 2$. So the number of those friends who are engaged in at least one activity is given by $|A \cup B \cup C|$ which by Theorem 4 is

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -|A \cap B| - |A \cap C| - |B \cap C| \\ &\quad +|A \cap B \cap C| \\ &= 6 + 6 + 6 \\ &\quad -4 - 2 - 3 \\ &\quad +2 \\ &= 11 \end{aligned}$$

And so $16 - 11 = 5$ of them engage in none of the activities.

3.3 Inclusion-Exclusion Principle (General Case)

Here is the nasty Inclusion-Exclusion formula for n sets. The formula is given both in words and in symbols. The word version is the one to remember, but look over the symbolic version to make sure that you really know what the theorem says.

⊙ **Theorem 5** (Inclusion-Exclusion for n sets)**.** *Let $A_1, A_2, A_3, \ldots, A_n$ be sets, not necessarily disjoint. Then the number of elements in their union is computed as follows:*

> add *the sizes of all individual sets* subtract *the sizes of all two-way intersection* add *the sizes of all three-way intersections* subtract *the sizes of all four-way intersections* add *the sizes of all ve-way intersections and so on.*

Equivalently, in symbols

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| &= \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| \\ &\quad - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n| \end{aligned} \tag{14}$$

We take the theorem without any proof.

***Example* 9. Counting Primes)** How many of the numbers 1, 2, . . . , 100 are prime?

Solution: One way to answer this question is to test each number up to 100 for primality and keep a count. This requires considerable effort. (Is 57 prime? How about 67?)

Another approach is to use the Inclusion-Exclusion Principle. However, to determine the number of primes, we will first count the number of non-primes. We can then find the number of primes by subtraction.

The set of non-primes in the range 1, . . . , 100 consists of the set C of composite numbers in this range (4, 6, 8, 9, . . . , 99, 100) and the number 1, which is neither prime nor composite. The main job is to determine the size of the set C of composite numbers. For this purpose, define:

$$A_p = \{x | 1 \le x \le 100,\ p|x, \text{ and } x \ne p\}$$

In words, A_p is the set of numbers in the range 1, . . . , 100 that are divisible by p, but not equal to p. For example, $A_2 = \{4, 6, 8, \ldots, 100\}$. We make the following claim:

Claim: $C = A_2 \cup A_3 \cup A_5 \cup A_7$

The claim explains the point of these funny A_p sets: we can write the set C of composite numbers as a union of them. We can then compute the number of elements in the union using Inclusion-Exclusion, and this will tell us the number of composite numbers in the range 1, . . . , 100.

Proof of the Claim: We prove the two sets equal by showing that each contains the other.
First, we show that $A_2 \cup A_3 \cup A_5 \cup A_7 \subset C$. Let a be an element of $A_2 \cup A_3 \cup A_5 \cup A_7$. Then for any one or more of $p = 2$, 3, 5, or 7, $a \in A_p$. This implies that a is in the range 1, . . . , 100, a is divisible by p, and a is not equal to p. This implies that a is a composite in the range 1, . . . , 100, and so $a \in C$.
Second, we show that $C \subset A_2 \cup A_3 \cup A_5 \cup A_7$. Let n be an element of C. Then n is a composite number in the range 1, . . . , 100. This means that n has at least two prime factors p and q. One of these must be 2, 3, 5, or 7. (Otherwise, both p and q are at least 11, and so $n \ge pq \ge 11 \cdot 11 = 121$, a contradiction.) This implies that n is an element of either A_2, A_3, A_5, or A_7, and so $n \in A_2 \cup A_3 \cup A_5 \cup A_7$.
As a result of equality of C and $A_2 \cup A_3 \cup A_5 \cup A_7$, we must have

$$|C| = |A_2 \cup A_3 \cup A_5 \cup A_7|$$

But this latter we can evaluate by the Inclusion-Exclusion. As a stepping stone we first calculate $|A_p|$ using the formula

$$|A_p| = \left[\frac{100}{p}\right] - 1$$

where $[x]$ represent the greatest integer less than of equal to x; whenever $n \le x < n+1$ for some integer n, then $[x] = n$. The first term, $\left[\dfrac{100}{p}\right]$, is the number of values in the range 1, ..., 100 that are divisible by p. The second term, 1, arises because we dened A_p to exclude p itself. This formula gives:

$$|A_2| = 49, \quad |A_3| = 32, \quad |A_5| = 19, \quad |A_7| = 13$$

Thus, the application of Inclusion-Exclusion gives

$$\begin{aligned} |C| &= |A_2 \cup A_3 \cup A_5 \cup A_7| \\ &= |A_2| + |A_3| + |A_5| + |A_7| \\ &\quad -|A_2 \cap A_3| - |A_2 \cap A_5| - |A_2 \cap A_7| - |A_3 \cap A_5| - |A_3 \cap A_7| - |A_5 \cap A_7| \\ &\quad +|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7| \\ &\quad -|A_2 \cap A_3 \cap A_5 \cap A_7| \end{aligned}$$

There are a lot of terms here! Fortunately, all of them are easy to evaluate. For example, $|A_3 \cap A_7|$ is the number of multiples of $3 \cdot 7 = 21$ in the range 1 to 100, which is $\left[\dfrac{100}{21}\right] = 4$. (Note that there is no reason to subtract 1 as we did when evaluating $|A_p|$ above.) Substituting values for all of the terms above gives:

$$\begin{aligned} |C| &= 49 + 32 + 19 + 13 \\ &\quad -16 - 10 - 7 - 6 - 4 - 2 \\ &\quad +3 + 2 + 1 + 0 \\ &\quad -0 \\ &= 74 \end{aligned}$$

This calculation shows that there are 74 composite numbers in the range 1 to 100. Since the number 1 is neither composite nor prime, there are $100 - 74 - 1 = 25$ primes in this range.
In retrospect, checking each number from 1 to 100 for primality and keeping a count of primes might have been easier! However, the Inclusion-Exclusion approach used here is asymptotically faster as the range of numbers grows large.

4 Recurrence Relations

The number of bacteria doubles every hour. If a colony starts with 5 bacteria, what will be the population of the colony after n hours. To answer the question, let a_n denote the population at the end of n hours. Since the number of bacteria doubles every hour, the relation $a_n = 2a_{n-1}$ holds whenever n is an integer. This relationship, together with the information that $a_0 = 5$ uniquely determines a_n for all non–negative integers n. We can find a formula for a_n from this information.

▷ **Definition 3** (Recurrence Relation). A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, ..., a_{n-1}, for all integers $n \ge n_0$, where n_0 is a non–negative integer. A sequence is called the *solution* of the recurrence relation if its terms satisfy the recurrence relation.

Usually, the goal is to find a *closed form* expression for the sequence; sometimes you want to find a specific value; occasionally, there's something else to do.

4.1 Modelling with recurrence relation

Recurrence relations can be used to model many problems. A few follow.

***Example* 10. Counting rabbits on an island.** Consider the following problem, which was originally proposed by Leonardo de Pisa, also known as Fibonacci, in the 13^{th} century in his book *Liber abaci.* A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are two months old, after which each pair produces another pair (again of opposite sex) each month. Find a recurrence relation for the number of *pairs* of rabbits after n months, assuming no rabbit dies.

Solution: Let f_n denote the number of rabbit pairs after n months. First note that $f_1 = f_2 = 1$ because a pair won't produce until it is two months old, so that at the end of two months there is only the original pair. For, $n \geq 3$, add the number of rabbits present at the end of $(n-1)$–st month and the newborn pairs. That means $f_n = f_{n-1} + f_{n-2}$ because only those pairs produce which are at least two months old. Hence, the recurrence relation becomes

$$f_n = f_{n-1} + f_{n-2}.$$

The sequence of numbers satisfying this recurrence is called *Fibonacci sequence.*

***Example* 11.** A sequence of n characters constructed from the elements of the set {0,1} is called *n–long* bit string. For example 10010 is 5–long bit string.

Find a recurrence relation and specify the initial conditions for the number of n–long bit strings with no two (or more) consecutive 0's. How many 5–long bit strings are there of this kind?
Solution: Assume $n \geq 3$ and let a_n denote the number of bit strings of length n that do not have two (or more) consecutive 0s. To obtain a recurrence relation for $\{a_n\}$, assume that we have obtained a sequence of $n-1$ bits. Now the n–th bit can be either a 0 *or* a 1. If the last bit is 0, then we must have a 1 in the $(n-1)$–th place *and* a sequence of $n-2$ bits satisfying the original requirement. Since there is only one way we can obtain this, the total number of such sequence which end with 0 becomes a_{n-2}. If the last bit is 1, we have a_{n-1} of them. Thus by the sum rule, the recurrence relation becomes

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 3\,.$$

The initial conditions are $a_1 = 2$, since both strings of length one, 0 and 1 do not have a consecutive 0s, and $a_2 = 3$ since in this case the valid strings are 01, 10, and 11.

To find a_5, we note first that $a_3 = a_2 + a_1 = 5$, $a_4 = a_3 + a_2 = 8$ and thus $a_5 = a_4 + a_3 = 8 + 5 = 13$. Also note that $\{a_n\}$ satisfies the same recurrence as the Fibonacci sequence.

***Example* 12. Codeword enumeration.** A computer system considers a string of decimal digits a valid codeword if it contains an even number of the digit 0. For example, 12409875608 is a valid codeword but 1230976800567 is not. Let a_n be the number of valid n digit codewords. Find a recurrence relation for a_n.
Solution: Note that $a_1 = 9$, since there are 10 one digit long strings out of which only one, namely, 0 is invalid. To obtain a recurrence for this sequence, we study how a valid code of n digit can be constructed out of a valid codeword having a $n-1$ digits. There are two ways to form a valid n digit codeword from a string with one less digit.

First, a valid codeword of n digits can be obtained by appending a valid code of $n-1$ digits with a digit except 0. This can be done in 9 ways. Hence, the number of n digits valid codewords obtained in this manner is $9a_{n-1}$.

Secondly, a valid codeword can be obtained by appending a zero at the end of an *invalid* $n-1$ digit codeword. The number of codewords of length n formed in this manner is equal to number of invalid codewords of length $n-1$. There are total of 10^{n-1} codewords of length $n-1$ out of which a_{n-1} are valid. Hence the number of invalid $n-1$ digits codes is $10^{n-1} - a_{n-1}$. And this same number of valid n digits codeword can be formed by appending a 0 at the end.

Since all valid n digit codewords are formed by one of these two methods, it follows that

$$\begin{aligned} a_n &= 9a_{n-1} + 10^{n-1} - a_{n-1} \\ &= 8a_{n-1} + 10^{n-1}. \end{aligned}$$

It turns out that in this case, the explicit form of a_n is:

$$a_n = \frac{1}{2}\left(2^{3n+1} - 8^n + 10^n\right)\,.$$

The next example establishes a recurrence that appear in many different contexts.

***Example* 13.** Find a recurrence relation for c_n, the number of ways to parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdot \cdots x_n$, to specify the order of multiplication. For example, $c_3 = 5$ since there are 5 ways to parenthesize the product $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ to determine the order of multiplication: $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, and $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$.
Solution: To solve the recurrence for c_n, we note that no matter how we put the brackets, one "·" operator,

namely that which specify the final multiplication, always remains out of these brackets. This final "·" operator appears between two of the $n+1$ numbers. Let us say that these two numbers are x_k and x_{k+1}.

Now, there are c_k number of ways to parenthesize the product of $k+1$ numbers: $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ *and* there are c_{n-k-1} ways to parenthesize the product of the remaining $n-k$ numbers: $x_{k+1} \cdot x_{k+2} \cdot x_{k+3} \cdot \cdots x_n$. Also, since each way of parenthesizing the first $k+1$ numbers can be associated with each way of parenthesizing the remaining $n-k$ numbers, there are $c_k c_{n-k-1}$ ways of parenthesizing the product of these $n+1$ numbers, when the final "·" operator appears between x_k and x_{k+1}. But since this final "·" operator can appear between any two pairs (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), it follows that

$$\begin{aligned} c_n &= c_0c_{n-1} + c_1c_{n-2} + c_2c_{n-3} + \cdots + c_{n-1}c_0 \\ &= \sum_{k=0}^{n-1} c_k c_{n-k-1} \end{aligned}$$

Note that the initial conditions are $c_0 = 1$ and $c_1 = 1$. This recurrence can be solved by the method of generating functions (discussed later) and it can be shown that

$$c_n = \frac{1}{n+1}\binom{2n}{n}. \tag{15}$$

The sequence $\{c_n\}$ of numbers, where c_n is given by (15) is known as the sequence of **Catalan Numbers**, and it arises in many contexts in combinatorics.

4.2 Solving recurrence relations

A wide variety of recurrence relations occur in mathematical models. Some of them can be solved by iterations or some *ad hoc* techniques. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are the recurrence relations that express the terms of the sequence as a *linear combination* of the previous terms.

▷ **Definition 4.** A *linear homogenous* recurrence relation of *degree k* with *constant coefficients* is a recurrence of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}, \tag{16}$$

where c_1, c_2, ..., c_k are constants belonging to the set of real numbers and $c_k \neq 0$.

The sequence $\{a_n\}$ is *uniquely* determined by the above recurrence relation and the k initial conditions:

$$a_0 = C_0,\ a_1 = C_1,\ \ldots,\ a_{k-1} = C_{k-1}.$$

For example $a_n = 1.11a_{n-1}$ is a linear homogenous recurrence relation of degree 1, $f_n = f_{n-1} + f_{n-2}$ is a linear homogenous recurrence relation of degree 2 and $a_n = a_{n-5}$ is a linear homogenous recurrence of degree 5. But the recurrence $h_n = 2h_{n-1} + 1$ is not homogenous while $a_n = a_{n-1} + a_{n-2}^2$ is not linear.

4.2.1 Solving linear homogeneous recurrence relations with constant coefficients

The basic approach in solving a linear homogeneous recurrence relation is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}.$$

When both sides of this equation is divided by r^{n-k} and the righthand side is subtracted throughout, we obtain an equivalent equation

$$r^k - c_1r^{k-1} - c_2r^{n-2} - \cdots - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation, which is called the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Let's concentrate on a linear homogeneous recurrence relation of the second degree. Further, suppose that the characteristic roots are *distinct*. The following theorem provides the solution of the recurrence relation:

⊙ **Theorem 6.** *Let c_1 and c_2 be some real constants. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants that are to be determined uniquely by the initial conditions*[1].

As it turns out that the above theorem is also applicable if the characteristic roots are complex numbers. Also the above theorem does not apply if the characteristic roots are coincident. For this case we have the following.

⊙ **Theorem 7.** *Let c_1 and c_2 be some real constants with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has a coincident root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.*

***Example* 14.** Find the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \qquad \text{with } a_0 = 2 \text{ and } a_1 = 7.$$

Solution: The characteristic equation is $r^2 - r - 2 = 0$ which has roots $r_1 = 2$ and $r_2 = -1$. Hence the sequence $\{a_n\}$ is a solution to the given recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2, \qquad \text{and} \quad a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these equations, we obtain $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the given recurrence with the initial conditions is

$$a_n = 3 \cdot 2^n - (-1)^n.$$

***Example* 15.** Find the solution of the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$.

Solution: The only root of the characteristic equation $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$, for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 1 = \alpha_1, \qquad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3,$$

from where it follows that $\alpha_0 = 1$ and $\alpha_1 = 1$. Thus the solution becomes

$$a_n = 3^n + n3^n.$$

The general result for the solution of a linear homogeneous recurrence relation of degree greater than two, but having *distinct characteristic roots* is as follows:

⊙ **Theorem 8.** *Let $c_1, c_2, \ldots, c_k$ be k real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

[1] Remember that a k degree recurrence relation is uniquely determined by k independent initial conditions.

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n,$$

where $n = 0, 1, 2, \ldots$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are constants.*

Following is an illustration.

***Example* 16.** Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3},$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.
Solution: The characteristic equation

$$r^3 - 6r^2 + 11r - 6 = 0$$

has roots $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$, since $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solution is of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_i's, use the initial conditions. Putting $n = 0, 1, 2$ in succession, we obtain

$$\begin{aligned} a_0 = 2 &= \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 = 5 &= \alpha_1 + 2\alpha_2 + 3\alpha_3, \\ a_2 = 15 &= \alpha_1 + 4\alpha_2 + 9\alpha_3. \end{aligned}$$

Solving these equations simultaneously, we find $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$.

Hence, the solution of the given recurrence with the given initial conditions is the sequence $\{a_n\}$, where

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Finally, we relax the condition of distinct roots and state the most general result for the solution of linear homogeneous recurrence relations. The key point is that *for each repeated root* r *of the characteristic equation, the general solution has a summand of the form* $P(n)r^n$*, where* $P(n)$ *is a polynomial in* n *of degree* $m - 1$ *with* m *the multiplicity of this root.*

⊙ **Theorem 9.** *Let* $c_1, c_2, \ldots, c_k$ *be* k *real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has t *distinct roots* $r_1, r_2, \ldots, r_t$ *with multiplicities* $m_1, m_2, \ldots, m_t$ *respectively, so that* $m_i \geq 1$ *for* $i = 1, 2, \ldots, t$ *and* $m_1 + m_2 + \cdots + m_t = k$*. Then a sequence* $\{a_n\}$ *is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2 + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \alpha_{2,2} n^2 + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \alpha_{t,2} n^2 + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$*, where* $\alpha_{i,j}$ *are constants for* $1 \leq i \leq t$ *and* $0 \leq j \leq m_i - 1$*.*

***Example* 17.** Find the solution of the recurrence

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3},$$

with the initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.
Solution: The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0,$$

which is the same as $(r+1)^3 = 0$. Thus the only root is -1 with multiplicity 3. The general solution is thus

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n.$$

To determine the constants $\alpha_{1,0}$, $\alpha_{1,1}$ and $\alpha_{1,2}$, use the initial conditions. Thus we have

$$\begin{aligned} a_0 = 1 &= \alpha_{1,0} \\ a_1 = -2 &= -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\ a_2 = -1 &= \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \end{aligned}$$

Solving, we obtain $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Thus the particular solution is the sequence $\{a_n\}$, where

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

5 Counting number of Integral Solutions of Linear equations

Suppose that 5 new academic positions are to be awarded to 4 departments in the university, with the restriction that no department is to be awarded more than 3 such positions, and that the Mathematics Department is to be awarded at least 1. We would like to nd out in how many ways this could be achieved. If we denote the departments by M, P, C, E, where M denotes the Mathematics Department, and denote by x_M, x_P, x_C, x_E the number of positions awarded to these departments respectively. Then clearly we must have

$$x_M + x_P + x_C + x_E = 5 \tag{17}$$

Furthermore

$$x_M \in \{1,\ 2,\ 3\} \quad \text{and} \quad x_P,\ x_C,\ x_E \in \{0,\ 1,\ 2,\ 3\} \tag{18}$$

We therefore need to find the number of solutions of the Eq. 17, subject to the restrictions (18). In general, we would like to find the number of solutions of an equation of the kind

$$x_1 + x_2 + x_3 + \ldots + x_k = n$$

where $n, k \in \mathbb{N}$, the set of natural numbers, are given and where the variables $x_1,\ x_2,\ x_3,\ \ldots,\ x_k$ are to assume integer values, subject to certain given restrictions.

5.1 Case A: The Simplest Case

Suppose that we are interested in finding the number of solutions of an equation of the type

$$x_1 + x_2 + x_3 + \ldots + x_k = n \tag{19}$$

where $n, k \in \mathbb{N}$ are given and the variables $x_1,\ x_2,\ \ldots x_n \in \{0,\ 1,\ 2,\ 3,\ \ldots\}$. It basically means we are trying to find the number of *non-negative* integral solutions of Eq. 19.

⊙ **Theorem 10** (No. of Non-Negative Integral solutions). *The number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_k = n$ is given by*

$$^{n+k-1}\mathrm{C}_n = {}^{n+k-1}\mathrm{C}_{k-1} = \frac{(n+k-1)!}{n!(k-1)!} \tag{20}$$

Proof. Consider a row of $(n+k-1)$ "+" signs as shown in the picture below:

$$\underbrace{+++++++++++++++++\ldots+++++++++++++++++}_{n+k-1}$$

Let us choose n of these "+" signs and change them to 1's. Clearly there are exactly $(k-1)$ "+" signs remaining, the same number of + signs as in the equation (19). The new situation is shown in the picture below:

$$\underbrace{\underbrace{1\ldots1}_{x_1}+\underbrace{1\ldots1}_{x_2}+\ldots+\underbrace{1\ldots1}_{x_k}}_{k \text{ groups}} \tag{21}$$

For example, the picture

$$+1111++1+111+11+\ldots+1111111+1111+11$$

denotes the information

$$0+4+0+1+3+2+\ldots+7+4+2$$

(note that consecutive + signs indicate an empty block of 1's in between, and the + sign at the left-hand end indicates an empty block of 1's at the left-hand end; similarly a + sign at the right-hand end would indicate an empty block of 1's at the right-hand end). It follows that our choice of the n 1's corresponds to a non-negative solution of the equation (19). Conversely, any solution of the equation (19), subject to the restriction that each of the x_i is a non-negative integer, can be illustrated by a picture of the type (21), and so corresponds to a choice of the n 1's. Hence the number of non-negative solutions of the equation (19) is equal to the number of ways we can choose n objects out of $(n+k-1)$. Clearly this is given by the binomial coecient indicated. □

5.2 Case B: A Minor Irritation

We next consider the situation when the variables have non-standard lower restrictions. Suppose that we are interested in finding the number of integral solutions of an equation of the type

$$x_1+x_2+x_3+\ldots+x_k=n \tag{22}$$

under the condition

$$x_1 \geq p_1, \quad x_2 \geq p_2, \ldots, x_k \geq p_k \tag{23}$$

where $k, n \in \mathbb{N}$ and each of the p_i's $\in \mathbb{N} \cup \{0\}$ such that $p_1+p_2+\ldots+p_k \leq n$. The main idea is to define the following variables :

$$\begin{aligned} y_1 &= x_1-p_1, \text{ then } y_1 \geq 0 \\ y_2 &= x_2-p_2, \text{ then } y_2 \geq 0 \\ &\vdots \\ y_k &= x_k-p_k, \text{ then } y_k \geq 0 \end{aligned} \tag{24}$$

In terms of these new variables, equation (22) becomes

$$y_1+p_1+y_2+p_2+y_3+p_3+\ldots+y_k+p_k=n$$

that is

$$y_1+y_2+y_3+\ldots+y_k=n-(p_1+p_2+p_3+\ldots+p_k)$$

where the variables y_i's are now just non-negative. But the number of solutions for this new equation can be obtained using the Theorem 10 of Case A by replacing n with $n-(p_1+p_2+\ldots+p_k)$ i.e.

$$^{n-(p_1+p_2+\ldots+p_k)+k-1}\mathrm{C}_{n-(p_1+p_2+\ldots+p_k)} = {}^{n-(p_1+p_2+\ldots+p_k)+k-1}\mathrm{C}_{k-1}$$

5.3 Case C: Inclusion-Exclusion

We next consider the situation when the variables have upper restrictions. Suppose that we are interested in finding the number of integral solutions of an equation of the type

$$x_1 + x_2 + x_3 + \ldots + x_k = n \tag{25}$$

under the condition

$$0 \leq x_1 \leq m_1, \quad 0 \leq x_2 \leq m_2, \ldots, 0 \leq x_k \leq m_k \tag{26}$$

Our approach is to, first of all, relax the upper restrictions on the variables, and solve instead the Case A problem of finding the number of non-negative integral solutions of the equation (25). Among these solutions will be some which violate the upper restrictions on the variables x_1, x_2, ..., x_k as given in (26). we must remove these from the number of solutions of the relaxed equation.

For each $i = 1, 2, 3, \ldots, k$, define the set A_i as the set of those solutions of the relaxed equation in which $x_i > m_i$. Then we need to subtract $|A_1 \cup A_2 \cup \ldots \cup A_k|$ from the number of solutions of the relaxed equation. However, from the equation (14)

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_k| &= \sum_{1\leq i\leq k} |A_i| - \sum_{1\leq i<j\leq k} |A_i \cap A_j| + \sum_{1\leq i<j<l\leq k} |A_i \cap A_j \cap A_l| \\ &\quad - \ldots + (-1)^{k+1}|A_1 \cap A_2 \cap \ldots \cap A_k| \end{aligned} \tag{27}$$

In order to evaluate each term of the sum we proceed as follows. The set A_1 contains all those solutions of the relaxed equation in which $x_1 > m_1$ i.e. $|A_1|$ will be the number of integral solutions of the equation

$$x_1 + x_2 + \ldots + x_k = n$$

subject to the conditions $x_1 > m_1$, $x_2 \geq 0$, $x_3 \geq 0$, ..., $x_k \geq 0$.

We apply the technique of Case B and define $y_1 = x_1 - (m_1 + 1)$ and $y_i = x_i$ for $i = 2, \ldots, k$; then each of variable y_i is a non-negative integer while the last equation becomes

$$y_1 + m_1 + 1 + y_2 + y_3 + \ldots + y_k = n$$

which becomes $y_1 + y_2 + \ldots + y_k = n - (m_1 + 1)$ which has ${}^{n-(m_1+1)+k-1}\mathrm{C}_{k-1}$ non-negative integral solutions. Similar arguments can be apply for each of $|A_2|, |A_3|, \ldots, |A_k|$. And so

$$\sum_{1\leq i\leq k} |A_i| = \sum_{1\leq i\leq k} {}^{n-(m_1+1)+k-1}\mathrm{C}_{k-1}$$

Similarly, if we consider $|A_1 \cap A_2|$, it is the number of integral solutions of the equation

$$x_1 + x_2 + \ldots + x_k = n$$

subject to the conditions $x_1 > m_1$, $x_2 > m_2$, $x_3 \geq 0$, ..., $x_k \geq 0$. Again we define $y_1 = x_1 - (m_1 + 1)$, $y_2 = x_2 - (m_2 + 1)$ and $y_i = x_i$ for $i = 3, \ldots, k$; then each of variable y_i is a non-negative integer while the last equation becomes

$$y_1 + m_1 + 1 + y_2 + m_2 + 1 + y_3 + \ldots + y_k = n$$

which becomes $y_1 + y_2 + \ldots + y_k = n - (m_1 + 1) - (m_2 + 1)$ which has

$${}^{n-(m_1+1)-(m_2+1)+k-1}\mathrm{C}_{k-1}$$

non-negative integral solutions. Similar arguments apply for every pair of intersection, and thus we get

$$\sum_{1\leq i<j\leq k} |A_i \cap A_j| = \sum_{1\leq i<j\leq k} {}^{n-(m_i+1)-(m_j+1)+k-1}\mathrm{C}_{k-1}$$

Continuing in this manner, we ultimately get

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_k| &= \sum_{1\le i\le k} {}^{n-(m_1+1)+k-1}\mathrm{C}_{k-1} - \sum_{1\le i<j\le k} {}^{n-(m_i+1)-(m_j+1)+k-1}\mathrm{C}_{k-1} \\ &+ \sum_{1\le i<j<l\le k} {}^{n-(m_i+1)-(m_j+1)-(m_l+1)+k-1}\mathrm{C}_{k-1} \\ &- \ldots + (-1)^{k+1} \quad {}^{n-(m_1+1)-(m_2+1)\ldots-(m_k+1)+k-1}\mathrm{C}_{k-1}\end{aligned}$$

Subtracting this number from ${}^{n+k-1}\mathrm{C}_{k-1}$ gives the desired number of solutions. Hence, we have the following:

⊙ **Theorem 11.** *The number of integral solutions of the equation of type*

$$x_1 + x_2 + \ldots + x_k = n$$

where

$$0 \le x_1 \le m_1, \quad 0 \le x_2 \le m_2, \ldots, 0 \le x_k \le m_k$$

is given by

$$\begin{aligned}{}^{n+k-1}\mathrm{C}_{k-1} &- \sum_{1\le i\le k} {}^{n-(m_1+1)+k-1}\mathrm{C}_{k-1} + \sum_{1\le i<j\le k} {}^{n-(m_i+1)-(m_j+1)+k-1}\mathrm{C}_{k-1} \\ &- \sum_{1\le i<j<l\le k} {}^{n-(m_i+1)-(m_j+1)-(m_l+1)+k-1}\mathrm{C}_{k-1} \qquad (28) \\ &+ \ldots + (-1)^{k} \quad {}^{n-(m_1+1)-(m_2+1)\ldots-(m_k+1)+k-1}\mathrm{C}_{k-1}\end{aligned}$$

5.4 Case Z: A Major Irritation

Suppose that we are interested in finding the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 23$$

where the variables x_1, $x_2 \in \{1, 3, 5, 7, 9\}$ and x_3, $x_4 \in \{3, 6, 7, 8, 9, 10, 11, 12\}$. Then it is very difficult and complicated, though possible, to make our methods discussed earlier work. We therefore need a slightly better approach. This will be the method of Generating Functions, which we shall study after Binomial Theorem.