Assignment on Continuity

Legends:

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of natural numbers.

1. Find all points of continuity of f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \sin|x| & \text{if } x \text{ is rational.} \end{cases}$$

2. Determine the set of points of continuity of f defined by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

3. Study the continuity of:

(a) The *Riemann function* defined as

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0, \\ \dfrac{1}{q} & \text{if } x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, \text{ and} \\ & \quad p, q \text{ are coprime.} \end{cases}$$

(b) the function defined as

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is irrational or } x = 0, \\ \dfrac{qx}{q+1} & \text{if } x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, \text{ and} \\ & \quad p, q \text{ are coprime.} \end{cases}$$

4. Prove that if a function f is continuous on $[a, b]$, the $|f|$ is also continuous on the given interval. Show by an example that the converse in not true.

5. Determine all a_n and b_n for which the function defined by

$$f(x) = \begin{cases} a_n + \sin \pi x & \text{if } x \in [2n, 2n+1],\ n \in \mathbb{Z}, \\ b_n + \cos \pi x & \text{if } x \in (2n-1, 2n),\ n \in \mathbb{Z}, \end{cases}$$

is continuous on $\mathbb{R}$.

6. Let $f(x) = [x^2] \sin \pi x$ for $x \in \mathbb{R}$, where $[\cdot]$ denotes the greatest integer function. Study the continuity of f.

7. Let f be defined as

$$f(x) = [x] + (x - [x])^{[x]} \quad \text{for } x \geq \frac{1}{2}$$

Show that f is continuous and that it is strictly increasing on $[1, \infty)$.

8. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then it attains a maximum and a minimum value.

9. Study the continuity of the following functions and sketch their graphs:

(a) $f(x) = \lim\limits_{n\to\infty} \dfrac{n^x - n^{-x}}{n^x + n^{-x}}$, for $x \in \mathbb{R}$

(b) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^2 e^{nx} + x}{e^{nx} + 1}$, for $x \in \mathbb{R}$

(c) $f(x) = \lim\limits_{n\to\infty} \dfrac{\ln(e^n + x^n)}{n}$, for $x \geq 0$

(d) $f(x) = \lim\limits_{n\to+\infty} \sqrt[n]{4^n + x^{2n} + \dfrac{1}{x^{2n}}}$ where $x \neq 0$

(e) $f(x) = \lim\limits_{n\to\infty} \sqrt[2n]{\cos^{2n} x + \sin^{2n} x}$, for $x \in \mathbb{R}$

10. Give an example of a bounded function on $[0, 1]$ which achieves neither a minimum nor a maximum.

11. Give an example of a bounded function on $[0, 1]$ which does not achieve its minimum on an $[a, b] \subset [0, 1]$, where $a < b$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with two *incommensurable* periods T_1 and T_2; that is $\frac{T_1}{T_2}$ is irrational. Prove that f is a constant function. Give an example of a nonconstant periodic function with two incommensurable periods.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with positive fundamental periods T_1 and T_2, respectively. Prove that if $\frac{T_1}{T_2} \notin \mathbb{Q}$, then $h = f + g$ is not a periodic function.

14. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a *fixed point* in $[0, 1]$; that is, there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

15. Assume that $f, g : [a, b] \to \mathbb{R}$ are continuous and such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $x_0 \in (a, b)$ for which $f(x_0) = g(x_0)$.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that

$$f\left(x_0 + \frac{T}{2}\right) = f(x_0)$$

17. A function $f : (a,b) \to \mathbb{R}$ is continuous. Prove that, given $x_1, x_2, \ldots, x_n$ in (a,b), there exists $x_0 \in (a,b)$, such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$$

18. (a) Prove that the equation $(1-x)\cos x = \sin x$ has at least one solution in $(0,1)$.

(b) For a nonzero polynomial $P(x)$, show that the equation $|P(x)| = e^x$ has at least one solution.

19. For $a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n$, show that all roots of the polynomial

$$P(x) = \prod_{k=0}^{n}(x + a_k) + 2\prod_{k=0}^{n}(x + b_k), \quad x \in \mathbb{R}$$

are real.

20. Suppose that f and g have the intermediate value property on $[a,b]$. Must $f+g$ posses the intermediate value property on that interval?

21. Assume that f is continuous on $[0,2]$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0,2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$. Give a geometric interpretation of this fact.

22. Let f be continuous on $[0,2]$. Show that there are x_1 and x_2 in $[0,2]$ such that $x_2 - x_1 = 1$ and $f(x_2) - f(x_1) = \dfrac{f(2) - f(0)}{2}$.

23. For $n \in \mathbb{N}$, let f be continuous on $[0,n]$ such that $f(0) = f(n)$. Prove that there exist x_1 and x_2 in $[0,n]$ satisfying $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

24. For $n \in \mathbb{N}$, let f be continuous on $[0,n]$ such that $f(0) = f(n)$. Prove that the equation $f(x) = f(y)$ has at least n solutions with $x - y \in \mathbb{N}$.

25. Suppose that real continuous functions f and g defined on $\mathbb{R}$ commute; that is $f(g(x)) = g(f(x))$ for $x \in \mathbb{R}$. Prove that if the equation $f(f(x)) = g(g(x))$ has a solution, then the equation $f(x) = g(x)$ also has.

Show by example that the assumption of continuity of f and g cannot be omitted.

26. Prove that a continuous injective function $f : \mathbb{R} \to \mathbb{R}$ is either strictly decreasing or strictly increasing.

27. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous injective function. Prove that if there exists n such that the n-th composition of f is an identity, that is, $f^n(x) = x$ for all $x \in \mathbb{R}$, then

(a) $f(x) = x,\ x \in \mathbb{R}$, if f is strictly increasing,

(b) $f^2(x) = x,\ x \in \mathbb{R}$, if f is strictly decreasing.

The n-th composition $f^n(x)$ is recursively defined as $f^1(x) = f(x)$ and for each natural $n \geq 2$, $f^n(x) = f(f^{n-1}(x))$.

28. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(f(x)) = -x$, $x \in \mathbb{R}$. Show that f cannot be continuous.

29. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which have the intermediate value property and such that there exists $n \in \mathbb{N}$ for which the n-th composition $f^n(x) = -x$, $x \in \mathbb{R}$.

30. A continuous function $f : [0,1] \to \mathbb{R}$ attains each of its values finitely many times and $f(0) \neq f(1)$. Show that f attains at least one of its values an odd number of times.