Topic M–11–4

Binomial Theorem

Contents

1 Introduction

Let's start by multiplying out $(x+a)(x+b)(x+c)(x+d)$. By actual multiplication it is easy to check that

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d) &= x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2 \\ &\quad +(abc+abd+acd+bcd)x+abcd \end{aligned} \quad (1)$$

We may, however, write this result by a bit of observation. The complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. Examining the way in which the various partial products are taken, we observe:

(i) the term x^4 is formed by taking the letter x out of *each* of the factors.

(ii) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor.

(iii) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and *two* of the letters a, b, c, d out of the remaining factors.

(iv) the terms involving x are formed by taking x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors.

(v) the terms independent of x is the product of letters a, b, c, d.

***Example* 1.** Expand $(x-2)(x+3)(x-5)(x+9)$.
Solution: From (1), we obtain

$$\begin{aligned}(x-2)(x+3)(x-5)(x+9) &= x^4+(-2+3-5+9)x^3+(-6+10-18-15+27-45)x^2 \\ &\quad +(30-54+90-135)x+270 \\ &= x^4+5x^3-47x^2-69x+270\end{aligned}$$

***Example* 2.** Find the coefficient of x^3 in $(x-3)(x+5)(x-1)(x+2)(x-8)$.
Solution: The terms involving x^3 can be obtained by multiplying together the x in *any three* of the factors, and *two* of the numbers out of the remaining two factors; hence, the coefficient is equal to the sum of the product of the numbers -3, 5, -1, 2, -8 taken two at at a time.

$$\begin{aligned}\text{Thus, the required coefficient} &= -15+3-6+24-5+10-40-2+8-16 \\ &= -39\end{aligned}$$

If in (1) we substitute $b=c=d=a$, we obtain

$$(x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4$$

which is a special case of the more general result (1). This approach of deducing a special case from a general one is sometimes more easier than to find the special case directly. Next, we use this approach to prove the *Binomial theorem*.

2 Binomial Theorem: Positive integral index

2.1 Binomial theorem

We begin by stating the following theorem.

⊙ **Theorem 1 (Binomial Theorem).** *For a positive integer n, the following holds true:*

$$(x+a)^n = {}^n\mathrm{C}_0x^n + {}^n\mathrm{C}_1x^{n-1}a + {}^n\mathrm{C}_2x^{n-2}a^2 + \ldots + {}^n\mathrm{C}_{n-1}xa^{n-1} + {}^n\mathrm{C}_na^n \quad (2)$$

Proof. Consider the expression

$$(x + a_1)(x + a_2)(x + a_3) \cdots (x + a_n)$$

the number of factors being n.

The expansion of this expression is the continued product of the n factors $(x + a_1)$, $(x + a_2)$, and so on till $(x + a_n)$, and every term in the expansion is of degree n in the sense that it is the product of n terms, *one* taken from each factor.
The highest power of x is x^n, and is formed by taking x from *each* of the n factors.
The terms involving x^{n-1} are formed by taking x out of *any* $n - 1$ of the factors, and *one* of the letters $a_1, a_2, a_3, \ldots, a_n$ from the remaining factor; thus the coefficient of x^{n-1} in the final product is the sum of $a_1, a_2, a_3, \ldots, a_n$; denote it by S_1.

The terms involving x^{n-2} are formed by taking x from *any* $n - 2$ factors, and *two* of the letters $a_1, a_2, a_3, \ldots, a_n$ from the two remaining factors; thus the coefficient of x^{n-2} in the final product is the sum of the products of $a_1, a_2, a_3, \ldots, a_n$ taken two at a time; denote it by S_2.

And, generally, the terms involving x^{n-r} are formed by taking x from *any* $n-r$ of the factors, and r of the letters $a_1, a_2, a_3, \ldots, a_n$ from the remaining r factors; thus the coefficient of x^{n-r} in the final product is the sum of the products of $a_1, a_2, a_3, \ldots, a_n$ taken r at a time; denote it by S_r.

The last term in the expansion is the product $a_1 a_2 a_3 \cdots a_n$; denote it by S_n.

Thus, we have

$$\begin{aligned}(x + a_1)(x + a_2)(x + a_3) \cdots (x + a_n) \;=\; & x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots \\ & + S_r x^{n-r} + \ldots + S_{n-1} x + S_n\end{aligned}$$

In S_1 the *number of terms* is n; in S_2 the *number of terms* is the same as the number of selections of 2 objects from a set of n, that is, ${}^n\mathrm{C}_2$; in S_3 the *number of terms* is ${}^n\mathrm{C}_3$; and so on.

Now, suppose $a_1 = a_2 = a_3 = \ldots = a_n = a$. Then S_1 becomes ${}^n\mathrm{C}_1 a$; S_2 becomes ${}^n\mathrm{C}_2 a^2$; and so on. Thus,

$$(x + a)^n = {}^n\mathrm{C}_0 x^n + {}^n\mathrm{C}_1 x^{n-1} a + {}^n\mathrm{C}_2 x^{n-2} a^2 + \ldots + {}^n\mathrm{C}_{n-1} x a^{n-1} + {}^n\mathrm{C}_n a^n$$

which completes the proof. □

♣ *Remark* 1. The numbers ${}^n\mathrm{C}_r$ are also called the *binomial numbers* because they occur in the binomial expansion.

♣ *Remark* 2. Note that the expansion on the right contains $n + 1$ terms.

♣ *Remark* 3. The binomial theorem might also be proved by using the principle of mathematical induction on n.

Replacing a by $-a$ in Eq. (1), we obtain the expansion of $(x - a)^n$:

$$(x - a)^n = {}^n\mathrm{C}_0 x^n - {}^n\mathrm{C}_1 x^{n-1} a + {}^n\mathrm{C}_2 x^{n-2} a^2 - {}^n\mathrm{C}_3 x^{n-3} a^3 + \ldots + (-1)^n\, {}^n\mathrm{C}_n a^n \qquad (3)$$

Thus, the terms in the expansion of $(x + a)^n$ and $(x - a)^n$ are *numerically* the same, but in $(x - a)^n$ they are alternately positive and negative, and the last term is positive if n is *even* and negative if n is *odd*.

***Example* 3.** Find the expansion of $(a - 2x)^7$.

$$(a - 2x)^7 = a^7 - {}^7\mathrm{C}_1 a^6 (2x) + {}^7\mathrm{C}_2 a^5 (2x)^2 - {}^7\mathrm{C}_3 a^4 (2x)^3 + \ldots \text{ to 8 terms.}$$

Now remembering the fact that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$, after calculating the coefficients up to ${}^7\mathrm{C}_3$, the rest may be written down at once; ${}^7\mathrm{C}_4 = {}^7\mathrm{C}_3$; ${}^7\mathrm{C}_5 = {}^7\mathrm{C}_2$; and so on. Thus, after calculating the values of ${}^7\mathrm{C}_r$ for $r = 0, 1, 2, \ldots, 7$, we get

$$(a - 2x)^7 = a^7 - 14a^6 x + 84a^5 x^2 - 280a^4 x^3 + 560a^3 x^4 - 672a^2 x^5 + 448a x^6 - 128x^7$$

The simplest form of the binomial theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Eq. (1) by replacing x by 1 and a by x. Thus,

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 x + {}^n\mathrm{C}_2 x^2 + {}^n\mathrm{C}_3 x^3 + \ldots + {}^n\mathrm{C}_{n-1} x^{n-1} + {}^n\mathrm{C}_n x^n \tag{4}$$

Further, replacing x by $-x$ in the above expression, we obtain the expansion of $(1-x)^n$:

$$(1-x)^n = {}^n\mathrm{C}_0 - {}^n\mathrm{C}_1 x + {}^n\mathrm{C}_2 x^2 - {}^n\mathrm{C}_3 x^3 + \ldots + (-1)^{n-1}\,{}^n\mathrm{C}_{n-1} x^{n-1} + (-1)^n\,{}^n\mathrm{C}_n x^n \tag{5}$$

2.2 General term

In the expansion of $(x+a)^n$, the coefficient of the second term is ${}^n\mathrm{C}_1$; of the third term is ${}^n\mathrm{C}_2$; of the fourth term is ${}^n\mathrm{C}_3$; and so on; the suffix in each term being one less than the number of the term to which it applies. Hence, ${}^n\mathrm{C}_r$ is the coefficient of the $(r+1)^{\text{th}}$ term. This term is called the **general term** of the expansion $(x+a)^n$ (denoted as T_{r+1}), because by giving to r different numerical values from 0 to n any of the coefficient can be found from ${}^n\mathrm{C}_r$; and by giving to x and a their appropriate indices any given term can be found. Thus, for the expansion of $(x+a)^n$, we get

$$\textbf{general term:} \qquad T_{r+1} = {}^n\mathrm{C}_r x^{n-r} a^r \tag{6}$$

In applying this result, it might be useful to remember that *the index of* a *is the same as the suffix of* C*, and that the sum of the indices of* x *and* a *is* n.

For the expansion of $(1+x)^n$, the general term becomes

$$\textbf{general term:} \qquad T_{r+1} = {}^n\mathrm{C}_r\, x^r \tag{7}$$

***Example* 4.** Find the fifth of $(a+2x^3)^{17}$.
Solution: The required term

$$\begin{aligned} T_5 = T_{4+1} &= {}^{17}\mathrm{C}_4 a^{13}(2x^3)^4 \\ &= \frac{17\cdot 16\cdot 15\cdot 14}{4\cdot 3\cdot 2\cdot 1} \times 16 a^{13} x^{12} \\ &= 38080\, a^{13} x^{12} \end{aligned}$$

***Example* 5.** Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

Solution: We have $(x^2-2x)^{10} = x^{20}\left(1-\dfrac{2}{x}\right)^{10}$, and since x^{20} multiplies every term in the expansion of $\left(1-\dfrac{2}{x}\right)^{10}$, we have to seek, in this expansion, the coefficient of the term that contains $\dfrac{1}{x^4}$.

Hence, the required coefficient is

$${}^{10}\mathrm{C}_4(-2)^4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} \times 16 = 3360$$

***Example* 6.** Find the coefficient of x^p in the expansion of $\left(x^2+\dfrac{1}{x^3}\right)^n$.

Solution: Suppose that x^p occurs in the $(r+1)^{\text{th}}$ term; that is, T_{r+1} contains x^p. Now,

$$T_{r+1} = {}^n\mathrm{C}_r (x^2)^{n-r}\left(\frac{1}{x^3}\right)^r = {}^n\mathrm{C}_r\, x^{2n-5r}$$

But this term contains x^p, and therefore we must have

$$2n - 5r = p \qquad \Rightarrow \quad r = \frac{2n-p}{5}$$

Thus, the required coefficient

$${}^n\mathrm{C}_r = {}^n\mathrm{C}_{(2n-p)/5} = \frac{n!}{\left(\dfrac{2n-p}{5}\right)!\,\left(\dfrac{3n+p}{5}\right)!}$$

Unless, $\dfrac{2n-p}{5}$ is a positive integer there will be no term containing x^p in the expansion.

2.3 Properties of Binomial numbers

Binomial numbers satisfy a number of useful properties. In this section, we investigate a few of them.

↬▶ **1.** ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$

Proof. Trivial □

↬▶ **2.** *Sum of the coefficients in the expansion of* $(1+x)^n$ *is* 2^n, *i.e.*

$$ {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_3 + \ldots + {}^n\mathrm{C}_n = 2^n \tag{8}$$

Proof. We know

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 x + {}^n\mathrm{C}_2 x^2 + {}^n\mathrm{C}_3 x^3 + \ldots + {}^n\mathrm{C}_n x^n$$

Setting $x = 1$ in this expression we get

$$2^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_3 + \ldots + {}^n\mathrm{C}_n$$

We can also prove the result combinatorially. We know that that 2^n is the total number of subsets (including the empty subset) of a set containing n distinct elements. On the other hand, we can also count the number of subsets in the following manner:

- number of empty sets = 1
- number of subsets containing exactly 1 element = ${}^n\mathrm{C}_1$.
- number of subsets containing exactly 2 elements = ${}^n\mathrm{C}_2$.
- number of subsets containing exactly 3 elements = ${}^n\mathrm{C}_3$.
- ......
- number of subsets containing exactly n elements = ${}^n\mathrm{C}_n$.

Adding them up we get Eq. (8). □

↬▶ **3.** *In the expansion of* $(1+x)^n$, *the sum of the odd coefficients equals the sum of the even coefficients, each being equal to* 2^{n-1}, *i.e.*

$$ {}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 + \ldots = {}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + {}^n\mathrm{C}_5 + \ldots = 2^{n-1} \tag{9}$$

Proof. We know

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 x + {}^n\mathrm{C}_2 x^2 + {}^n\mathrm{C}_3 x^3 + \ldots + {}^n\mathrm{C}_n x^n$$

Setting $x = -1$, we obtain

$$0 = {}^n\mathrm{C}_0 - {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 - {}^n\mathrm{C}_3 + {}^n\mathrm{C}_4 - \ldots + (-1)^n\,{}^n\mathrm{C}_n$$

Collecting all the even terms which have minus sign in front of them on one side, we obtain

$$ {}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 + \ldots = {}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + {}^n\mathrm{C}_5 + \ldots = S \quad \text{(say)}$$

From Eq.(8), we have

$$\begin{aligned} 2^n &= ({}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 + \ldots) + ({}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + {}^n\mathrm{C}_5 + \ldots) \\ &= S + S = 2S \\ \Rightarrow \quad S &= 2^{n-1} \end{aligned}$$

Thus, we get Eq. (9) □

↬▶ **4 (Pascal triangle).** *The following property leads to the construction of the* Pascal's triangle*:*

$$ {}^{n}\mathrm{C}_{r} + {}^{n}\mathrm{C}_{r+1} = {}^{n+1}\mathrm{C}_{r+1} \tag{10} $$

Proof. Suppose we want to construct a subset containing $r+1$ objects from a set of $n+1$ objects. For this purpose, we just have to select $r+1$ objects from the given set. This can be done in ${}^{n+1}\mathrm{C}_{r+1}$ ways and this number also becomes the number of such subsets.

On the other hand, we can construct those subsets when one particular object is always to be included in our subsets or when one particular object is always left out. In this case, we obtain the required number as ${}^{n}\mathrm{C}_{r} + {}^{n}\mathrm{C}_{r+1}$ and since these two ways of generating subsets gives the total number of subsets containing $r+1$ objects, we get Eq. (10). □

The above result can be displayed in the following diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & {}^{0}\mathrm{C}_{0} & & & & & \\
 & & & & {}^{1}\mathrm{C}_{0} & & {}^{1}\mathrm{C}_{1} & & & & \\
 & & & {}^{2}\mathrm{C}_{0} & & {}^{2}\mathrm{C}_{1} & & {}^{2}\mathrm{C}_{2} & & & \\
 & & {}^{3}\mathrm{C}_{0} & & {}^{3}\mathrm{C}_{1} & & {}^{3}\mathrm{C}_{2} & & {}^{3}\mathrm{C}_{3} & & \\
 & {}^{4}\mathrm{C}_{0} & & {}^{4}\mathrm{C}_{1} & & {}^{4}\mathrm{C}_{2} & & {}^{4}\mathrm{C}_{3} & & {}^{4}\mathrm{C}_{4} & \\
{}^{5}\mathrm{C}_{0} & & {}^{5}\mathrm{C}_{1} & & {}^{5}\mathrm{C}_{2} & & {}^{5}\mathrm{C}_{3} & & {}^{5}\mathrm{C}_{4} & & {}^{5}\mathrm{C}_{5}
\end{array}
$$

This pyramidal structure of binomial numbers shows the beginning of what is referred to as the *Pascal triangle*. Its construction utilizes Eq. (10) by keeping in mind that n represents a *row* while r specifies a *column* of a row. Thus Eq. (10), in words, state that *to get a element (except the leading element in each row) in this triangle, just add the two elements diagonally left and right up.*

We can replace each binomial number by its value to get another version of Pascal triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

In this form, it is quite easy to see that each element except the leading 1's, is the sum of the two numbers immediately above it. This property of the Pascal triangle enables us to generate the triangle very fast, building it up row by row. It also gives us a tool to prove many properties of the binomial coefficients. For example, let's ask: what is the sum of *squares* of elements in each row?

Let's experiment, by computing the sum of squares of elements in the the first few rows:

$$
\begin{aligned}
1^2 &= 1, \\
1^2 + 1^2 &= 2, \\
1^2 + 2^2 + 1^2 &= 6, \\
1^2 + 3^2 + 3^2 + 1^2 &= 20, \\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70
\end{aligned}
$$

We may recognize these numbers as the numbers in the middle column of the Pascal triangle. Of course, only every second row contains an entry in the middle column, so the last value above, the sum of squares in row No. 4, is the middle element in row No. 8. So the examples above suggest the following identity:

↬▶ **5.** *The binomial numbers satisfy*

$$ {}^n\mathrm{C}_0{}^2 + {}^n\mathrm{C}_1{}^2 + {}^n\mathrm{C}_2{}^2 + {}^n\mathrm{C}_3{}^2 + \ldots + {}^n\mathrm{C}_n{}^2 = {}^{2n}\mathrm{C}_n \tag{11} $$

Proof. Of course, the few experiments above do not prove that this identity always holds, so we need a proof. We will give an interpretation of both sides of the identity as the result of a counting problem; it will turn out that they count the same things, so they are equal. It is obvious what the right hand side counts: the number of subsets of size n of a set of size $2n$. It will be convenient to choose, as our $2n$–element set, the set $S = \{1, 2, 3, \ldots, 2n\}$.

The combinatorial interpretation of the left hand side is not so easy. Consider a typical term, say ${}^n\mathrm{C}_r{}^2$. We claim that this is the number of those n-element subsets of S that contain exactly r elements from the set $\{1, 2, 3, \ldots, n\}$ (the first half of our set). In fact, how do we choose such an n-element subset of S? We choose the first r elements from $\{1, 2, \ldots, n\}$ and the remaining $n - r$ from $\{n + 1, n + 2, \ldots, 2n\}$. The first part can be done in ${}^n\mathrm{C}_r$ ways; no matter which r-element subset of $\{1, 2, \ldots, n\}$ we selected, we have ${}^n\mathrm{C}_{n-r}$ to choose the other part. Thus the number of ways to choose an n-element subset of S having r elements from $\{1, 2, \ldots, n\}$ is

$$ {}^n\mathrm{C}_r\,{}^n\mathrm{C}_{n-r} = {}^n\mathrm{C}_r{}^2 \qquad \text{since } {}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r} $$

Now to get the total number number of n-element subsets of S, we have to sum these numbers for all values of $k = 0, 1, 2, \ldots, n$. This proves Eq. (11). □

To get another proof of the above property, expand $(1 + x)^{2n}$ in two ways: first, straight forward apply binomial expansion; second write $(1+x)^{2n} = (1+x)^n(1+x)^n$, expand each term, multiply out term by term and look for the coefficient of x^n in both cases — first and second.

I mention the following two properties of binomial numbers (without proof), which you might try to prove as exercises.

↬▶ **6** (**Vondermonde's identity**).

$$ {}^n\mathrm{C}_0\,{}^m\mathrm{C}_k + {}^n\mathrm{C}_1\,{}^m\mathrm{C}_{k-1} + {}^n\mathrm{C}_2\,{}^m\mathrm{C}_{k-2} + \ldots + {}^n\mathrm{C}_k\,{}^m\mathrm{C}_0 = {}^{n+m}\mathrm{C}_k \tag{12} $$

↬▶ **7.**

$$ {}^n\mathrm{C}_0 + {}^{n+1}\mathrm{C}_1 + {}^{n+2}\mathrm{C}_2 + {}^{n+3}\mathrm{C}_3 + \ldots + {}^{n+k}\mathrm{C}_k = {}^{n+k+1}\mathrm{C}_k \tag{13} $$

2.4 Greatest term of the expansion

Suppose we try to find the *numerically* greatest term in the expansion of $(x + a)^n$. For this purpose, let's pretend that we *have* already found the greatest term and it is the $(r + 1)^{\text{th}}$ term of the expansion. Since T_{r+1} is *numerically* the greatest term, the ratio of $|T_{r+1}|$ with $|T_r|$ must be either more than 1 or equal to it (in case, there are two terms that are numerically greater than the other ones). Thus, we must have

$$ \left|\frac{T_{r+1}}{T_r}\right| \geq 1 $$

and we must select the *greatest integral value of* r consistent with this inequality. Since $T_{r+1} = {}^n\mathrm{C}_r x^{n-r} a^r$ and $T_r = {}^n\mathrm{C}_{r-1} x^{n-r+1} a^{r-1}$, we obtain

$$ \left|\frac{T_{r+1}}{T_r}\right| \geq 1 \qquad \Rightarrow \left|\frac{{}^n\mathrm{C}_r x^{n-r} a^r}{{}^n\mathrm{C}_{r-1} x^{n-r+1} a^{r-1}}\right| \geq 1 \qquad \Rightarrow \left|\frac{n-r+1}{r}\cdot\frac{a}{x}\right| \geq 1 $$

The last inequality finally simplifies to

$$ \left|\frac{n+1}{r} - 1\right| \geq \left|\frac{x}{a}\right| \qquad \Rightarrow \left|\frac{n+1}{r} - 1\right| \geq \frac{|x|}{|a|} $$

Now since r can only vary between 0 and $n+1$, the ratio $(n+1)/r$ is always more than or equal to 1. As such the left side of the previous inequality reduces to $\dfrac{n+1}{r} - 1$ and we obtain

$$\frac{n+1}{r} - 1 \geq \frac{|x|}{|a|} \qquad \Rightarrow \ r \leq \frac{(n+1)|a|}{|x|+|a|}$$

We must select the integral value of r consistent with this inequality. To that end, we just set r as the greatest integer contained in the right side of the last inequality. Thus, the required value of r is given by

$$r = \left[\frac{(n+1)|a|}{|x|+|a|}\right] \tag{14}$$

where $[\alpha]$ represents the *greatest integer less than or equal to* α. Specifically, if α is a real number such that, for an integer N,

$$N \leq \alpha < N+1 \qquad \text{then} \qquad [\alpha] = N \tag{15}$$

Once the value of r is determined, the numerically greatest term is the $(r+1)^{\text{th}}$ term, i.e. T_{r+1}.

***Example* 7.** If $x = \frac{1}{3}$, find the greatest term in the expansion of $(1+4x)^8$.
Solution: Using Eq. 14 with $n = 8$, $x = 1$, and $a = 4x = \frac{4}{3}$, we obtain

$$r = \left[\frac{9 \times \frac{4}{3}}{1 + \frac{4}{3}}\right] = \left[\frac{36}{7}\right] = 5$$

Thus, the greatest term is the sixth one, and its value is

$${}^8\mathrm{C}_5 \left(\frac{4}{3}\right)^5 = \frac{57344}{243}$$

***Example* 8.** Find the greatest term in the expansion of $(3-2x)^9$ when $x = 1$.
Solution: Use of Eq. 14 with $n = 9$, $x = 3$ and $a = -2x = -2$ yields the value of r as

$$r = \left[\frac{10 \times 2}{3+2}\right] = 4$$

Thus, the numerically greatest term is the fifth one, its value being ${}^9\mathrm{C}_4 \times 3^5 \times (-2)^4 = 489888$.

3 Binomial Theorem: Any index

So far we have investigated the Binomial theorem when the index is any positive integer; we shall now consider the case when the index is *any real number*. Since any binomial can always be reduced to the expansion of $(1+x)^n$, we take only this expansion.

Before stating the general result, first note the following circumstances. Suppose we have two expressions arranged in ascending powers of x, such as

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \tag{16}$$

$$\text{and} \qquad 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \tag{17}$$

The product of these two expressions will be a series in ascending powers of x; let say that it is

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \ldots$$

It is quite obvious that the coefficients A, B, C, D, ... are functions of m and n, and therefore the actual values of these coefficients for a given case will depend on the values of m and n in that case. But the way in which the coefficients of the powers of x in (16) and (17) combine to give A, B, C, ... is quite independent of m and n. In other words, *whatever values* m *and* n *might have,* A, B, C, ... *preserve the same invariable form.* If therefore we can determine the form of A, B, C, ... for any particular values of m and n, we conclude that A, B, C, ... will have the *same* form *for all values* of m and n.

We make use of this idea to construct a general proof of binomial theorem for any index. (The original idea is due to Euler.) But before that, the result follows:

⊙ **Theorem 2.** *For a variable x satisfying $|x| < 1$ and a* real *number α, the following holds true:*

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \ldots \tag{18}$$

3.1 Binomial theorem: positive fraction

For any value whatsoever of α, let us denote by $f(\alpha)$ the following expression

$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \ldots$$

Then

$$f(m) = 1 + mx + \frac{m(m-1)}{2!}\,x^2 + \frac{m(m-1)(m-2)}{3!}\,x^3 + \ldots$$

$$\text{and} \qquad f(n) = 1 + nx + \frac{n(n-1)}{2!}\,x^2 + \frac{n(n-1)(n-2)}{3!}\,x^3 + \ldots$$

If we multiply these two series, the product will be another series in ascending powers of x, whose *coefficients will be unaltered in form whatever m and n may be.*

To determine this *invariable form of the product* we may give to m and n any values that are most convenient. For this purpose, let m and n take values that are positive integers. In this case $f(m)$ coincides with the expanded form of $(1+x)^m$ which we have already studied. Similarly, $f(n)$ is the expanded form of $(1+x)^n$. Therefore,

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n}$$

but when m and n are positive integers, the expansion of $(1+x)^{m+n}$ is

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{2!}\,x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{3!}\,x^3 + + \ldots$$

where I have used the expanded form of Binomial numbers ${}^n\mathrm{C}_r$. This, then, is the *form* of the product of $f(m) \times f(n)$ *in all cases*, whatever the values of m and n may be; and in agreement with our notation, we denote the above series as $f(m+n)$; therefore *for all values of m and n*, we conclude that

$$f(m) \times f(n) = f(m+n)$$

Similarly, we can prove that

$$f(m) \times f(n) \times f(s) = f(m+n) \times f(s) = f(m+n+s)$$

Proceeding in this manner we can show that (with slight change in notations)

$$f(m_1) \times f(m_2) \times f(m_3) \times \ldots \times f(m_k) = f(m_1 + m_2 + m_3 + \ldots m_k)$$

Now, if we set $m_1 = m_2 = m_3 = \ldots = m_k = \dfrac{p}{k}$, where p is a positive integer, the last equality gives us

$$\left\{ f\left(\frac{p}{k}\right) \right\}^k = f(p)$$

But since p is a positive integer, $f(p) = (1+x)^p$. Thus, we obtain

$$\left\{ f\left(\frac{p}{k}\right) \right\}^k = (1+x)^p \qquad \Rightarrow \; f\left(\frac{p}{k}\right) = (1+x)^{p/k}$$

But, by assumption, $f\left(\dfrac{p}{k}\right)$ stands for

$$1 + \frac{p}{k}\,x + \frac{\frac{p}{k}\left(\frac{p}{k}-1\right)}{2!}\,x^2 + \frac{\frac{p}{k}\left(\frac{p}{k}-1\right)\left(\frac{p}{k}-2\right)}{3!}\,x^3 + \ldots$$

Thus, we obtain,

$$(1+x)^{p/k} = 1 + \frac{p}{k}\,x + \frac{\frac{p}{k}\left(\frac{p}{k}-1\right)}{2!}\,x^2 + \frac{\frac{p}{k}\left(\frac{p}{k}-1\right)\left(\frac{p}{k}-2\right)}{3!}\,x^3 + \ldots \qquad (19)$$

which proves the Binomial theorem for any positive fractional index.

3.2 Binomial theorem: negative index

We have already proved that

$$f(m) \times f(n) = f(m+n)$$

for any values of m and n. Replacing m by $-n$ (where n is positive), we have

$$f(-n) \times f(n) = f(-n+n) = f(0) = 1$$

since all terms of the series except the first vanish. Therefore,

$$\frac{1}{f(n)} = f(-n)$$

But $f(n) = (1+x)^n$ for *any positive* value, we get

$$\frac{1}{(1+x)^n} = f(-n) \qquad \Rightarrow \; (1+x)^{-n} = f(-n)$$

But $f(-n)$ stands for

$$1 + (-n)x + \frac{(-n)(-n-1)}{2!}\,x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}\,x^3 + \ldots$$

Hence, we obtain

$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}\,x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}\,x^3 + \ldots \qquad (20)$$

which proves the Binomial theorem for any negative index. Hence, in all cases, we proved that

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \ldots$$

and the Binomial theorem is established for all real α.

♣ *Remark* 4. Since for an arbitrary index α, the coefficients of the powers of x in the expansion of $(1+x)^\alpha$ cannot be expressed via the binomial numbers, the general term in this case must be written in full:

$$T_{r+1} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-r+1)}{r!}\,x^r \tag{21}$$

Also the coefficient of the general term can never vanish unless one of the factors of its numerator is zero; the series will therefore stop at the $(r+1)^{\text{th}}$ term, when $n-r+1$ is zero; that is when $r = n+1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus *the expansion by Binomial theorem extends to $n+1$ terms only, when n is a positive integer, and to an infinite number of terms in all other cases.* As such, *convergence* (which we shall not take up for the moment) is an issue for an arbitrary α. We shall for the moment note that the series expansion of $(1+x)^\alpha$ for any real α is valid only when $|x| < 1$.

♣ *Remark* 5. For a negative *integer* $-n$ (where n is positive integer), the expansion of $(1+x)^{-n}$ can be written in terms of the binomial numbers (also shown is T_{r+1}):

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + \frac{(-n)(-n-1)}{2!}\,x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}\,x^3 + \dots \\ &= 1 - nx + \frac{n(n+1)}{2!}\,x^2 - \frac{n(n+1)(n+2)}{3!}\,x^3 + \dots \\ &= 1 - {}^{n}\mathrm{C}_1\,x + {}^{n+1}\mathrm{C}_2\,x^2 + {}^{n+2}\mathrm{C}_3\,x^3 + \dots + (-1)^r\,{}^{n+r-1}\mathrm{C}_r\,x^r + \dots \end{aligned} \tag{22}$$

***Example* 9.** Expand $(1-x)^{3/2}$ to four terms.

$$\begin{aligned}(1-x)^{3/2} &= 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}(-x)^3 + \dots \\ &= 1 - \frac{3}{2}\,x + \frac{3}{8}\,x^2 + \frac{1}{16}\,x^3 + \dots\end{aligned}$$

***Example* 10.** Find the general term in the expansion of $(2+3x)^{-4}$.

Solution: Since $(2+3x)^{-4} = 2^{-4}\left(1+\frac{3x}{2}\right)^{-4}$, applying Eq. 22, we obtain the general term as

$$T_{r+1} = 2^{-4}(-1)^r\;{}^{3+r}\mathrm{C}_r\left(\frac{3x}{2}\right)^r = {}^{3+r}\mathrm{C}_r(-1)^r\,\frac{3^r}{16\times 2^r}\,x^r$$

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