Solutions to Assignment M–11–3

(Complex Numbers – 2)

1. The main idea is to make the denominator real (usually) by multiplying with the conjugate of the denominator in the numerator and denominator.
(i) Transform as follows:

$$\frac{a+i\,b}{c+i\,d}=\frac{(a+i\,b)(c-i\,d)}{(c+i\,d)(c-i\,d)}=\frac{(ac+bd)+i\,(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\,\frac{bc-ad}{c^2+d^2}$$

(ii) Notice that $(1+i\,)^2=1+i^{\,2}+2i\;=1-1+2i\;=2i$. So

$$\frac{(1+i\,)^2}{1-i}=\frac{2i}{1-i}=\frac{2i(1+i)}{(1-i)(1+i)}=\frac{2i(1+i)}{2}=i(1+i)=-1+i$$

(iii) Since

$$\frac{1+i}{1-i}=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{2i}{2}=i$$

Therefore, $\left(\dfrac{1+i}{1-i}\right)^3=i^3=-i.$
(iv) The given expression is same as

$$\frac{(x+i\,y)^2-(x-i\,y)^2}{x^2+y^2}=\frac{4i\,xy}{x^2+y^2}$$

(v) $(8-i\,\sqrt{6})/14$

2. (i) $e^{i\,\pi/2}$ (ii) $\sqrt{2}\;e^{i\,\pi/4}$ (iii) $2\;e^{i\,\pi}$ (iv) $3\;e^{-i\,\pi/2}$ (v) $2\sqrt{3}\;e^{i\,\pi/3}$

3. (i) Since $8+6i=9-1+6i=3^2+i^2+2\cdot3\cdot i=(3+i)^2$. Hence $\sqrt{8+6i}=\pm(3+i)$.
(ii) $-1+2\sqrt{-2}=-1+2i\sqrt{2}=-(1-2i\sqrt{2})=-(2-1-2i\sqrt{2})=-\big((\sqrt{2})^2+i^2-2\cdot\sqrt{2}\cdot i\big)=-(\sqrt{2}-i)^2=i^2(\sqrt{2}-i)^2=(1+i\,\sqrt{2})^2$.
Therefore, $\sqrt{-1+2\sqrt{-2}}=\pm(1+i\,\sqrt{2})$
(iii) $21-20i=25-4-20i=5^2+(2i)^2-2\cdot5\cdot2i=(5-2i)^2$. So $\sqrt{21-20i}=\pm(5-2i)$.
(iv) $a+i\,\sqrt{1-a^2}=\dfrac{1}{2}\left(2a+2i\,\sqrt{1-a^2}\right)=\dfrac{1}{2}\left((1+a)-(1-a)+2i\,\sqrt{(1-a)(1+a)}\right)=\dfrac{1}{2}\left(\sqrt{1+a}+i\,\sqrt{1-a}\right)^2$
So, $\sqrt{a+i\,\sqrt{1-a^2}}=\pm\dfrac{1}{\sqrt{2}}\left(\sqrt{1+a}+i\,\sqrt{1-a}\right)$
(v) Notice that $x^4+x^2+1=x^4+2x^2+1-x^2=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$. And so the given expression can be transformed as

$$\frac{1}{2}\left(2x+2i\sqrt{(x^2+x+1)(x^2-x+1)}\right)=\frac{1}{2}\left((x^2+x+1)-(x^2-x+1)+2\cdot\sqrt{x^2+x+1}\cdot\sqrt{x^2-x+1}\right)$$

whence the given expression equals $\dfrac{1}{2}\left(\sqrt{x^2+x+1}+i\,\sqrt{x^2-x+1}\right)^2$. Hence,

$$\sqrt{x+i\,\sqrt{x^4+x^2+1}}=\pm\frac{1}{\sqrt{2}}\left(\sqrt{x^2+x+1}+i\,\sqrt{x^2-x+1}\right).$$

(vi) $2i=1-1+2i=1+i^2+2i=(1+i)^2$. So, $\sqrt{2i}=\pm(1+i)$.
(vii) $\dfrac{2-36i}{2+3i}=-8-6i=-(8+6i)=-(3+i)^2=i^2(3+i)^2=(-1+3i)^2$. Therefore, $\sqrt{\dfrac{2-36i}{2+3i}}=\pm(-1+3i)$
(viii) We have $x^2+\dfrac{1}{x^2}-\dfrac{4}{i}\left(x+\dfrac{1}{x}\right)-2=x^2+\dfrac{1}{x^2}+2-4+4i\left(x+\dfrac{1}{x}\right)=\left(x+\dfrac{1}{x}\right)^2+(2i)^2+2\cdot2i\cdot\left(x+\dfrac{1}{x}\right)=\left(x+\dfrac{1}{x}+2i\right)^2$. So $\sqrt{x^2+\dfrac{1}{x^2}-\dfrac{4}{i}\left(x+\dfrac{1}{x}\right)-2}=\pm\left(x+\dfrac{1}{x}+2i\right)$.

4. Since $x=3+2i$, so $(x-3)^2=(2i)^2$ which gives $x^2-6x+13=0$. Next we notice that $x^4-4x^3+4x^2+8x+39=(x^2-6x+13)(x^2+2x+3)$ and hence the result follows. Next, if $y=\overline{x}$, then $x^2+xy+y^2=x^2+x\overline{x}+\overline{x}^2=x^2+\overline{x}^2+|x|^2=(x+\overline{x})^2-2x\overline{x}+|x|^2=(2\mathrm{Re}\,(x))^2-|x|^2$. Since $\mathrm{Re}\,(x)=3$ and $|x|^2=13$, we get $(2\mathrm{Re}\,(x))^2-|x|^2=36-13=23$. (The problem wrongly asks to prove that $x^2+xy+y^2=0$.)

5. Since $\dfrac{(1+4i)+(3+10i)}{2} = 2+7i$. As such the point $2+7i$ is the mid-point of the line joining $1+4i$ and $3+10i$. As such the three points must be collinear.

6. (i) Define $z_1 = a + i\,b$, $z_2 = c + i\,d$, $z_3 = e + i\,f$. Then $(a^2+b^2)(c^2+d^2)(e^2+f^2) = |z_1|^2|z_2|^2|z_3|^2$. But since $|z_1|^2|z_2|^2|z_3|^2 = |z_1z_2z_3|^2$, we must have

$$(a^2+b^2)(c^2+d^2)(e^2+f^2) = |(a+i\,b)(c+i\,d)(e+i\,f)|^2 = |(ace-bde-adf-bcf)+i(acf-bdf+ade+bce)|^2$$

And hence $(a^2+b^2)(c^2+d^2)(e^2+f^2) = (ace-bde-adf-bcf)^2 + (acf-bdf+ade+bce)^2$.
(ii) Similar to (i).

7. Let $z_1 = a + i\,b$, $z_2 = c + i\,d$. Then $|z_1|^2 = a^2+b^2$ and $|z_2|^2 = c^2+d^2$. Also $z_1\overline{z}_2 = (ac+bd)+i(bc-ad)$ implying that $2(ac+bd) = 2\mathrm{Re}\,(z_1\overline{z}_2) = z_1\overline{z}_2 + \overline{z}_1z_2$. Also been given $x/y = z_1/z_2 \Rightarrow xz_2 = yz_1$.
The LHS of the given expression

$$\begin{aligned}(c^2+d^2)x^2 - 2(ac+bd)xy + (a^2+b^2)y^2 &= |z_2|^2x^2 - (z_1\overline{z}_2 + \overline{z}_1z_2)xy + |z_1|^2y^2 \\ &= z_2\overline{z}_2x^2 - z_1\overline{z}_2xy - \overline{z}_1z_2xy + z_1\overline{z}_1y^2 \\ &= \overline{z}_2x(z_2x - z_1y) - \overline{z}_1y(z_2x - z_1y) \\ &= (z_2x - z_1y)(\overline{z}_2x - \overline{z}_1y) \\ &= 0 = \mathrm{RHS}\end{aligned}$$

8. Since by Euler's formula, $\cos x + i\,\sin x = e^{ix}$, so the LHS of the given expression becomes

$$\prod_{k=1}^{n}(\cos k\theta + i\,\sin k\theta) = \prod_{k=1}^{n} e^{i\,k\theta} = e^{i\,\theta}e^{i\,2\theta}\cdots e^{i\,n\theta} = e^{i\,(1+2+\ldots+n)\theta} = e^{i\,n(n+1)\theta/2}$$

Also $1 = e^{i\,0}$. Hence the given equation becomes $e^{i\,n(n+1)\theta/2} = e^{i\,0}$. By equality of complex numbers in the exponential form, we get $n(n+1)\theta/2 = 2m\pi$ where $m \in \mathbb{I}$, the set of integers. As such $\theta = 4m\pi/n(n+1)$ for $m \in \mathbb{I}$.

9. Since $x + \dfrac{1}{x} = 2\cos\varphi$, rearranging we get $x^2 - 2x\cos\varphi + 1 = 0$. Solving we get $x = e^{\pm i\,\varphi}$. Similarly $y = e^{\pm i\,\theta}$. So one of the values of xy is $e^{i\,(\varphi+\theta)}$. Since $|xy| = |e^{i\,(\varphi+\theta)}| = 1$, so $1/xy = \overline{xy}$ and hence $xy + \dfrac{1}{xy} = xy + \overline{xy} = 2\mathrm{Re}\,(xy) = 2\cos(\varphi+\theta)$. So that one value of the given expression is $\cos(\varphi+\theta)$.
Also $x^m = e^{\pm i\,m\varphi}$ and $y^n = e^{\pm i\,n\theta}$ so one of the values of $\dfrac{x^m}{y^n}$ is $e^{i\,(m\varphi-n\theta)}$ and that of $\dfrac{y^n}{x^m}$ is $e^{i\,(n\theta-m\varphi)} = e^{-i\,(m\varphi-n\theta)}$. Hence $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m} = 2\cos(m\varphi - n\theta)$.

10. Using Euler's formula, we have $\cos\dfrac{\pi}{2^r} + i\,\sin\dfrac{\pi}{2^r} = e^{i\,\pi/2^r}$. So the given product is same as

$$e^{i\,\pi/2}e^{i\,\pi/2^2}e^{i\,\pi/2^3}\cdots = \exp\left(i\,\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right)\right) = \exp\left(i\,\frac{\pi}{2}\frac{1}{(1-1/2)}\right) = e^{i\,\pi} = -1$$

11. (i) Use the fact that $\arg(z_1z_2\cdots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$.
(ii) Use the fact that $|z_1|^2|z_2|^2\cdots|z_n|^2 = |z_1z_2\cdots z_n|^2$.

12. The given equation $x^2 - 2x\cos\varphi + 1 = 0$ has as roots the numbers $e^{i\,\varphi}$ and $e^{-i\,\varphi}$. Hence, we require an equation whose roots are $e^{i\,n\varphi}$ and $e^{-i\,n\varphi}$ in which case the sum of roots $= e^{i\,n\varphi} + e^{-i\,n\varphi} = 2\cos n\varphi$ and the product $= e^{i\,n\varphi}e^{-i\,n\varphi} = 1$. And as such the required equation is $x^2 - 2x\cos n\varphi + 1 = 0$.

13. If $(1+x)^n = p_0 + p_1x + p_2x^2 + \cdots$, show that

$$p_0 - p_2 + p_4 - \cdots = 2^{n/2}\cos\tfrac{1}{4}n\pi \qquad \text{and} \qquad p_1 - p_3 + p_5 - \cdots = 2^{n/2}\sin\tfrac{1}{4}n\pi$$

Set $x = i$ and use the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ to get

$$(1+i)^n = p_0 + i\,p_1 - a_2 - i\,p_3 + p_4 + i\,p_5 - p_6 - i\,p_7 + p_8 + \ldots$$

Since $(1+i) = \sqrt{2}\,(\cos(\pi/4) + i\,\sin(\pi/4))$, using de'Moivre's formula, we get

$$(1+i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\,\sin\frac{\pi}{4}\right)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\,\sin\frac{n\pi}{4}\right)$$

And so

$$2^{n/2}\left(\cos\frac{n\pi}{4}+i\,\sin\frac{n\pi}{4}\right)=p_0+i\,p_1-p_2-i\,p_3+p_4+i\,p_5-p_6-i\,p_7+p_8+\ldots$$

That is to say (after collecting the real and imaginary parts)

$$2^{n/2}\left(\cos\frac{n\pi}{4}+i\,\sin\frac{n\pi}{4}\right)=(p_0-p_2+p_4-p_6+\ldots)+i\,(p_1-p_3+p_5-p_7+\ldots)$$

Comparing the real and imaginary parts, we get the desired result.

14. We can write $a+i\,b=re^{i\,\varphi}$, where $r=\sqrt{a^2+b^2}$ and $tan\varphi=b/a$. Since $a-i\,b$ is the complex conjugate of $a+i\,b$, so $a-i\,b=re^{-i\,\varphi}$. Therefore

$$\sqrt[n]{a+i\,b}=\sqrt[n]{r}\,\exp\left(i\,\frac{2k\pi+\varphi}{n}\right),\qquad k=0,1,2,\ldots,(n-1)$$

And similarly

$$\sqrt[n]{a-i\,b}=\sqrt[n]{r}\,\exp\left(i\,\frac{2m\pi-\varphi}{n}\right),\qquad m=0,1,2,\ldots,(n-1)$$

For every value of $k=0$ and $m=0$, the value of $\sqrt[n]{a+i\,b}$ is $\sqrt[n]{r}\,e^{i\,\varphi/n}$ and that of $\sqrt[n]{a-i\,b}$ is $\sqrt[n]{r}\,e^{-i\,\varphi/n}$ the two of which add to a real value. For every other k choose the value of $m=n-k$. Since $k\in\{1,2,\ldots,(n-1)\}$ so will $m=n-k\in\{1,2,\ldots,(n-1)\}$. For these values of k and m, the numbers $\sqrt[n]{r}\,\exp\left(i\,\frac{2k\pi+\varphi}{n}\right)$ and $\sqrt[n]{r}\,\exp\left(i\,\frac{2m\pi-\varphi}{n}\right)$ are conjugates, since

$$\exp\left(i\,\frac{2k\pi+\varphi}{n}\right)\times\exp\left(i\,\frac{2m\pi-\varphi}{n}\right)=\exp\left(i\,\frac{2\pi(k+m)}{n}\right)=\exp\left(i\,\frac{2\pi n}{n}\right)=e^{i\,2\pi}=1$$

and hence their sum will be purely real. Hence, we get a total of n distinct real values for the sum $\sqrt[n]{a+i\,b}+\sqrt[n]{a-i\,b}$, which are $\sqrt[n]{r}$ times the numbers $2\cos\left(\frac{2k\pi+\varphi}{n}\right)$ for $k=0,\,1,\,2,\,\ldots,\,(n-1)$.

15. (i) Geometrically $|a-b|$ for any two complex numbers a and b represent the distance between the corresponding points on the Argand plane. The equation $|z-2+3i|=2$ can be rewritten as $|z-(2-3i)|=2$. So the locus of z consists of all points in the Argand plane whose distance from the point $2-3i$ is 2 units. But that gives a circle centred at the point $2-3i$ and having radius $=2$.
(ii) For the same reason as in (i), the condition $|z+2i|\le 1$ represents a circular region (including the perimeter) centred at the point $-2i$ and having a radius of 1 unit.
(iii) Notice that $\mathrm{Re}(\bar{z}+i\,)=\mathrm{Re}\,(\bar{z})=\mathrm{Re}\,(z)$. And therefore the given condition is the same as $\mathrm{Re}\,(z)=4$, but that is the straight line corresponding to $x=4$ is the Cartesian plane.
(iv) The given condition is same as $|z-(1-2i)|=|z-(-3-i)|$ which says that z should be equidistant from the two fixed points $1-2i$ and $-3-i$ implying that the locus of z will be the straight line that bisects the segment joining $1-2i$ and $-3-i$.
(v) The given equation is same as $|z-(-i)|+|z-i|=4$, i.e. the distance of z from the points i and $-i$ is a constant $=4$ which is greater than 2 (which is the distance between i and $-i$). The required curve is an ellipse centred at the origin having the points i and $-i$ as the focii.
(vi) Since the distance between the points $2i$ and $-2i$ is already 4 so the required locus consists of all the points lying on the imaginary axis excluding the segment joining $-2i$ and $2i$.

16. The given equation is equivalent to $|z_1-z_2|=|z_1+z_2|$. Since $z_2\neq 0$, we can divide throughout by $|z_2|$ to obtain $\left|\frac{z_1}{z_2}-1\right|=\left|\frac{z_1}{z_2}+1\right|$. That is, the complex number $\frac{z_1}{z_2}$ is equidistant from the point $1+0i$ and $-1+0i$. So z_1/z_2 must lie on the perpendicular bisector of the segment joining these two points $1+0i$ and $-1+0i$.

17. Equivalently, we need to prove $|1-z_1\bar{z}_2|^2-|z_1-z_2|^2<0$. Since $|z|^2=z\bar{z}$, the left hand side is

$$\begin{aligned}(1-z_1\bar{z}_2)(1-\bar{z}_1z_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)&\\ &= 1-\bar{z}_1z_2-z_1\bar{z}_2+|z_1|^2|z_2|^2-|z_1|^2+z_1\bar{z}_2+\bar{z}_1z_2-|z_2|^2\\ &= 1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2\\ &= (1-|z_1|)(1-|z_2|)<0\end{aligned}$$

since $1-|z_1|>0$ and $1-|z_2|<0$.

18. The given equation is same as $(z^p-1)(z^q-1)=0$ which in turn imply $z^p-1=0$ or $z^q-1=0$. So if α is a root of the given equation then either $\alpha^p=1$ or $\alpha^q=1$ which means α is either a p-th root of unity or a q-th root. And so either $1+\alpha+\alpha^2+\ldots+\alpha^{p-1}=0$ or $1+\alpha+\alpha^2+\ldots+\alpha^{q-1}=0$.
To prove that both of them cannot simultaneously hold true, lets assume to the contrary that actually hold simultaneously. Then α must be simulatneously be a p-th as well as a q-th root of unity. The p-th roots of unity are $e^{i\,2k\pi/p}$ for $k=0,\,1,\,2,\,\ldots,\,p-1$ while the q-th roots are $e^{i\,2m\pi/q}$ for $m=0,\,1,\,2,\,\ldots,\,q-1$. If α is at the same time a p-th and a q-th root of unity then there should be some values of k and m satisfying $k\in\{0,1,2,\ldots,p-1\}$ and $m\in\{0,1,2,\ldots,q-1\}$ so that $e^{i\,2k\pi/p}=e^{i\,2m\pi/q}$ i.e. to say that $\exp\left(i\,2\pi\left(\frac{k}{p}-\frac{m}{q}\right)\right)=1$ implying therefore that $\frac{k}{p}-\frac{m}{q}$ must be an integer (since $e^{i2\pi\times\text{an integer}}=1$). Say $\frac{k}{p}-\frac{m}{q}=n$ for some integer n. This gives $kq=p(nq+m)$. We see that p divides the RHS so it must divide the LHS but that is not possible as $k<p$ while q is prime to p. And hence we cannot have $kq=p(nq+m)$ contradicting the statement that $\frac{k}{p}-\frac{m}{q}$ is an integer, which in turn means that α cannot at the same time be a p-th as well as a q-th root of unity.

19. We get

$$\left|\frac{z+1}{z-1}\right|=\frac{|z+1|}{|z-1|}=\frac{\sqrt{(3+t)^2+3-t^2}}{\sqrt{(1+t)^2+3-t^2}}=\sqrt{3}$$

after a bit of simplification. If we take $z=x+i\,y$, $(x,y\in\mathbb{R}$, then we get $x=2+t$ and $y=\sqrt{3-t^2}$. Obviously, for y to be real we must have $3\geq t^2$ i.e. $-\sqrt{3}\leq t\leq\sqrt{3}$. We then get $(x-2)^2=t^2$ and $3-y^2=t^2$, implying therefore the locus as part of the curve given by $(x-2)^2+y^2=3$ which is actually a circle centred at the point $2+i\,0$ and having a radius of $\sqrt{3}$ units.

20. Use the fact that $|z_1\pm z_2|^2=|z_1|^2+|z_2|^2\pm 2\text{Re}\,(z_1\bar{z}_2)$. Geometrically, this equation reflects the property of a parallelogram that the sum of the squares of the diagonals is equal to the sum of the squares of the sides.

21. This is a consequence of the *triangle's inequality*

$$|z_1+z_2+\ldots+z_n|\leq|z_1|+|z_2|+\ldots+|z_n|$$

Use this inequality on the complex numbers $z_k=a_k+i\,b_k$, $(k=1,\,2,\,\ldots,\,n)$ to get the desired result.

22. This is equivalent to $\arg\left(\frac{3(z-(2+i\,))}{2(z-(4+3i))}\right)=\frac{\pi}{4}$. Since $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$, so the above is

$$\arg\left(\frac{3}{2}\right)+\arg\left(\frac{z-(2+i\,)}{z-(4+3i)}\right)=\frac{\pi}{4}\quad\Rightarrow\arg\left(\frac{z-(2+i\,)}{z-(4+3i)}\right)=\frac{\pi}{4}$$

since $\arg(3/2)=0$. This means that the line joining the (variable) complex point $P(z)$ to $A(2+i\,)$ and the line joining z to $B(4+3i)$ have a constant angle angle. But we know that (only) in a circle a chord subtends the same angle at any point on the perimeter. So the required locus must be a part of the circle in which the segment AB is a chord subtending an angle of $\pi/4$ at the perimeter and so subtending a right angle at the center. If we draw lines through the points A and B parallel to the real and imaginary axis, their intersection gives the possible location of the center. In this case, we get two possibilities: the point $(2+3i)$ or the point $(4+i)$. Since the angle between the two lines must be measured (positive) between the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ (after making them co-terminal) from the vector in the denominator to the numerator, we can eliminate one of the circles. In this given problem the locus will be the major arc AB of the circle centred at $4+i$.

23. Solve for z: (i) $2|z|^2+z^2-5+i\sqrt{3}=0$, (ii) $z^3+\overline{z}=0$.
(i) The equation could be solved by taking $z=a+i\,b$. But we take an alternative approach. The given equation is $2|z|^2+z^2-5+i\sqrt{3}=0$ i.e.

$$2|z^2|-5=z^2-i\sqrt{3}\tag{1}$$

Since the left side is purely real, the right side must be. So the imaginary part of z^2 must be $\sqrt{3}$. So z^2 is of the form $z^2=x+i\sqrt{3}$ for some real number x. Then $|z^2|=\sqrt{x^2+3}$. And hence Equation 1 becomes

$$\begin{aligned}2\sqrt{x^2+3}-5=x\quad&\Rightarrow\quad 4(x^2+3)=(x+5)^2\\&\Rightarrow\quad 3x^2-10x-13=0\\&\Rightarrow\quad x=-1,\ 13/3\end{aligned}$$

Therefore, $z^2 = -1 + i\sqrt{3}$ or $z^2 = \frac{13}{3} + i\sqrt{3}$. If

$$z^2 = -1 + i\sqrt{3} = \frac{-2 + 2i\sqrt{3}}{2} = \frac{1 + (i\sqrt{3})^2 + 2i\sqrt{3}}{2} = \frac{(1 + i\sqrt{3})^2}{2}$$

then $\boxed{z = \pm\frac{1 + i\sqrt{3}}{\sqrt{2}}}$. On the other hand, if

$$z^2 = \frac{13}{3} + i\sqrt{3} = \frac{13 + 3i\sqrt{3}}{3} = \frac{26 + 2i\,3\sqrt{3}}{6} = \frac{(3\sqrt{3})^2 + i^2 + 2i\,3\sqrt{3}}{6} = \frac{(3\sqrt{3} + i)^2}{6}$$

Hence, $\boxed{z = \pm\frac{3\sqrt{3} + i}{\sqrt{6}}}$.

(ii) $z^3 + \overline{z} = 0 \Rightarrow z^3 = -\overline{z}$. If two complex numbers are equal, then so are their modulii. Hence, we get

$$\begin{aligned} |z^3| = |-\overline{z}| \quad &\Rightarrow \quad |z|^3 = |z| \qquad (\because |\overline{z}| = |z|) \\ &\Rightarrow \quad |z|(|z|^2 - 1) = 0 \\ &\Rightarrow \quad |z| = 0, \text{or } |z| = 1 \end{aligned}$$

If $|z| = 0 \Rightarrow \boxed{z = 0}$. If $|z| = 1$, multiplying the original equation with $\overline{z}$, we get $z^4 + |z|^2 = 0$ i.e. $z^4 = -1$ (since $|z| = 1$). This means that the z will be the forth roots of -1. i.e. $z = \boxed{\frac{1+i}{\sqrt{2}}}, \boxed{\frac{-1+i}{\sqrt{2}}}, \boxed{\frac{-1-i}{\sqrt{2}}}, \boxed{\frac{1-i}{\sqrt{2}}}$.

24. Since for cube roots of unity ω and ω^2, we have the factorization

$$x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y)$$

so the given equation is

$$(z + ab)^3 - a^3 = 0 \Rightarrow (z + ab - a)(z + ab - \omega a)(z + ab - \omega^2 a) = 0$$

So that the roots are $z_1 = a(1-b)$, $z_2 = a(\omega - b)$ and $z_3 = a(\omega^2 - b)$. The pairwise distance between these points: $|z_1 - z_2| = |a||1 - \omega| = |a|\sqrt{3}$, $|z_2 - z_3| = |a||\omega - \omega^2| = |a|\sqrt{3}$, and $|z_3 - z_1| = |a||\omega^2 - 1| = |a|\sqrt{3}$ are all equal. And hence they must be the vertices of an equilateral triangle. The side length of this triangle is $|a|\sqrt{3}$ and hence the area is $\frac{\sqrt{3}}{4}(|a|\sqrt{3})^2 = \frac{3\sqrt{3}}{4}|a|^2$.

25. Since $|z| = 1$, the locus of z is the unit circle. Let $|1 - z| = t$ so that $t \in [0, 2]$ as t represents the distance of z from the point 1. Also then

$$t^2 = |1 - z|^2 = (1 - z)(1 - \overline{z}) = 1 - z - \overline{z} + z\overline{z} = 1 - (z + \overline{z}) + |z|^2 = 2 - 2\text{Re}(z)$$

so that $2\text{Re}(z) = 2 - t^2$. Also

$$|1 - z + z^2|^2 = (1 - z + z^2)(1 - \overline{z} + \overline{z}^2) = 1 - \overline{z} + \overline{z}^2 - z + z\overline{z} - z\overline{z}^2 + z^2 - z^2\overline{z} + z^2\overline{z}^2$$

Since $z\overline{z}^2 = z\overline{z}\overline{z} = |z|^2\overline{z} = \overline{z}$, and similarly $z^2\overline{z} = z$, and $z^2 + \overline{z}^2 = (z + \overline{z})^2 - 2z\overline{z} = (z + \overline{z})^2 - 2$, the above expression can be simplified to

$$|1 - z + z^2|^2 = 1 - 2(z + \overline{z}) + (z + \overline{z})^2 = (1 - (z + \overline{z}))^2 = (1 - 2\text{Re}(z))^2 = (t^2 - 1)^2$$

so that $|1 - z + z^2| = |t^2 - 1|$. So the given expression

$$|1 - z| + |1 - z + z^2| = t + |t^2 - 1| = f(t) \qquad \text{for } t \in [0, 2]$$

It is easy to check (for example, by drawing a graph of $y = f(t)$) that the minimum value $\boxed{f_{\min} = 1}$ attained at $t = 0$ or 1 meaning for $z = 1$ or $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ while the maximum value $\boxed{f_{\max} = 5}$ attained for $t = 2$ meaning for $z = -1$.

26. (i) centroid $z_G = \dfrac{z_1+z_2+z_3}{2}$
(ii) orthocenter $z_H = \dfrac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A+\tan B+\tan C}$
(iii) incenter $z_I = \dfrac{z_1|z_2-z_3| + z_2|z_3-z_1| + z_3|z_1-z_2|}{|z_2-z_3|+|z_3-z_1|+|z_1-z_2|}$
(iv) circumcenter $z_S = \dfrac{z_1\sin 2A + z_2\sin 2B + z_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}$. Using the determinant notation, it can be expressed as

$$z_S = \frac{\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ |z_1|^2 & |z_2|^2 & |z_3|^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \overline{z}_1 & \overline{z}_2 & \overline{z}_3 \end{vmatrix}}$$

And the area

$$\Delta = \text{Absolute value of } \frac{i}{4}\begin{vmatrix} z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \\ z_3 & \overline{z}_3 & 1 \end{vmatrix} = \text{Absolute value of } \frac{1}{2}\text{Im}(\overline{z}_1 z_2 + \overline{z}_2 z_3 + \overline{z}_3 z_1)$$

27. From the condition $|z_1-a| = |z_2-a| = |z_3-a|$, it is quite clear that a is equidistant from each of the vertices z_1, z_2, and z_3 of the triangle. As such a must represent the circumcenter of the triangle. But since the triangle is equilateral, the circumcenter coincides with the centroid, and hence

$$a = \frac{z_1+z_2+z_3}{3} \quad \Rightarrow z_1+z_2+z_3 = 3a$$

28. We have $z_1+z_2 = -p$, and $z_1z_2 = q$, and by geometry $z_2 = z_1e^{i\alpha}$. Therefore, we have $q = z_1z_2 = z_1^2e^{i\alpha}$ which gives $z_1^2 = qe^{-i\alpha}$. And so $z_2^2 = z_1^2e^{-2i\alpha} = qe^{i\alpha}$. Also,

$$\begin{aligned} p^2 &= (-p)^2 = (z_1+z_2)^2 = z_1^2+z_2^2+2z_1z_2 \\ &= qe^{-i\alpha} + qe^{i\alpha} + 2q = q(e^{i\alpha}+e^{-i\alpha}) + 2q = 2q\cos\alpha + 2q \\ &= 2q(1+\cos\alpha) \end{aligned}$$

which completes the required proof.

29. If $a = 0$, the given equation reduces to $2|z|+1 = 0$, which cannot be true for any complex number since $|z| \ge 0$, so that $2|z|+1 > 0$ for every complex number. So we need consider only $a > 0$.

Rewriting the equation as $2|z|+1 = a(4z-i)$, we notice that the left side of the equality is a purely real number and so the right side must be purely real. Writing $z = x+iy$ $(x, y \in \mathbb{R})$, we see that $4(x+iy)-i = 4x+i(4y-1)$ must be real, so that $y = 1/4$. With this condition, the original equation gives (upon squaring)

$$4|z|^2 = (4ax-1)^2 \quad \Rightarrow 4\left(x^2+\frac{1}{16}\right) = 16a^2x^2 - 8ax + 1$$

which gives upon simplification

$$16x^2(4a^2-1) - 32ax + 3 = 0$$

If $a = \frac{1}{2}$, then $4a^2-1 = 0$, and the equation then gives $x = \dfrac{3}{16}$.

For every other positive a, we get upon solving:

$$x = \frac{4a \pm \sqrt{4a^2+3}}{4(4a^2-1)}$$

Hence, the following are the solutions:

- $z = \dfrac{3}{16} + \dfrac{i}{4}$ for $a = \dfrac{1}{2}$,
- $z = \dfrac{4a\pm\sqrt{4a^2+3}}{4(4a^2-1)} + \dfrac{i}{4}$ for $a \in (0, 1/2) \cup (1/2, \infty)$

30. With the substitution $n-k=r$, the given sum becomes

$$S=\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n}=\sum_{r=1}^{n-1}r\cos\left(\frac{2\pi}{n}(n-r)\right)=\sum_{r=1}^{n-1}r\cos\left(2\pi-\frac{2r\pi}{n}\right)=\sum_{r=1}^{n-1}r\cos\frac{2r\pi}{n}$$

But $\cos\theta=\mathrm{Re}(e^{i\theta})$, so the given sum

$$S=\sum_{r=1}^{n-1}r\mathrm{Re}(e^{i2r\pi/n})=\mathrm{Re}\left(\sum_{r=1}^{n-1}re^{i2r\pi/n}\right)=\mathrm{Re}\left(\sum_{r=1}^{n-1}r(e^{i2\pi/n})^r\right)=\mathrm{Re}\left(\sum_{r=1}^{n-1}r\alpha^r\right)$$

where $\alpha=e^{i2\pi/n}$. Now consider

$$J=\sum_{r=1}^{n-1}r\alpha^r=\alpha+2\alpha^2+3\alpha^3+\ldots+(n-1)\alpha^{n-1}$$

Multiplying this equation with α, we get

$$\alpha J=\alpha^2+2\alpha^3+3\alpha^4+\ldots+(n-2)\alpha^{n-1}+(n-1)\alpha^n$$

Therefore, subtracting we get

$$(1-\alpha)J=(\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{n-1})-(n-1)\alpha^n$$

Since $\alpha=e^{i2\pi/n}$, therefore $\alpha^n=e^{i2\pi}=1$. So

$$\begin{aligned}(1-\alpha)J &= (\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{n-1})-(n-1)\\ &= 1+\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{n-1}-n\\ &= \frac{1-\alpha^n}{1-\alpha}-n\\ &= -n \qquad (\because \alpha^n=1)\\ \therefore J &= \frac{-n}{1-\alpha}\end{aligned}$$

However,

$$1-\alpha=1-e^{i2\pi/n}=\left(1-\cos\frac{2\pi}{n}\right)-i\sin\frac{2\pi}{n}=2\sin^2\frac{\pi}{n}-i2\sin\frac{\pi}{n}\cos\frac{\pi}{n}=2\sin\frac{\pi}{n}\left(\sin\frac{\pi}{n}-i\cos\frac{\pi}{n}\right)$$

Therefore,

$$J=\frac{-n}{1-\alpha}=\frac{-n}{2\sin\frac{\pi}{n}\left(\sin\frac{\pi}{n}-i\cos\frac{\pi}{n}\right)}=\frac{-n}{2\sin\frac{\pi}{n}}\left(\sin\frac{\pi}{n}+i\cos\frac{\pi}{n}\right)$$

Hence, $S=\mathrm{Re}\ J=-n/2$. Incidentally, we also get (by equating the imaginary parts):

$$\sum_{k=1}^{n-1}(n-k)\sin\frac{2k\pi}{n}=-\frac{n}{2}\ \cot\frac{\pi}{n}$$

31. $z^{10}-1=0\rightarrow\ z^{10}=1$. Therefore, the roots of the given equation are simply the 10th roots of unity, i.e. $z_k=e^{i2k\pi/10}$ for $k=0,1,\ldots,9$. Hence, we can factorize

$$z^{10}-1=(z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-z_7)(z-z_8)(z-z_9) \tag{2}$$

Next, we notice that $z_0=1$, $z_5=e^{i\pi}=-1$. Also

$$z_6=e^{i12\pi/10}=e^{i(20-8)\pi/10}=e^{i(2\pi-(8\pi/10))}=e^{-8\pi/10}=\overline{z_4}$$

Similarly, $z_7=\overline{z_3}$, $z_8=\overline{z_2}$, and $z_9=\overline{z_1}$, so that we can write Equation 2 as

$$z^{10}-1=(z-1)(z+1)\{(z-z_1)(z-\overline{z_1})\}\{(z-z_2)(z-\overline{z_2})\}\{(z-z_3)(z-\overline{z_3})\}\{(z-z_4)(z-\overline{z_4})\} \tag{3}$$

Next, we multiply each pair of factors containing the conjugate roots to get (for $k=1,2,3,4$) quadratic factors as follows:

$$(z-z_k)(z-\overline{z_k})=z^2-z(z_k+\overline{z_k})+z_k\overline{z_k}=z^2-2z\cos\frac{2k\pi}{10}+|z_k|^2=z^2-2z\cos\frac{k\pi}{5}+1$$

Using this, Equation 3 becomes

$$z^{10}-1=(z^2-1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right) \quad (4)$$

Divide both sides by z^5 to get

$$z^5-\frac{1}{z^5}=\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}-2\cos\frac{\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{2\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{3\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{4\pi}{5}\right) \quad (5)$$

Since $\cos\frac{3\pi}{5}=\cos\left(\pi-\frac{2\pi}{5}\right)=-\cos\frac{2\pi}{5}$ and similarly $\cos\frac{4\pi}{5}=-\cos\frac{\pi}{5}$. Therefore, we get from Equation 5

$$\begin{aligned} z^5-\frac{1}{z^5} &= \left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}-2\cos\frac{\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{2\pi}{5}\right)\left(z+\frac{1}{z}+2\cos\frac{2\pi}{5}\right)\left(z+\frac{1}{z}+2\cos\frac{\pi}{5}\right) \\ &= \left(z-\frac{1}{z}\right)\left(\left(z+\frac{1}{z}\right)^2-4\cos^2\frac{\pi}{5}\right)\left(\left(z+\frac{1}{z}\right)^2-4\cos^2\frac{2\pi}{5}\right) \end{aligned}$$

Now set $z=e^{i\varphi}$, so that $z^5=e^{i5\varphi}$, $\frac{1}{z}=e^{-i\varphi}$, and $\frac{1}{z^5}=e^{-i5\varphi}$ and therefore $z^5-\frac{1}{z^5}=2i\sin 5\varphi$, and $z-\frac{1}{z}=2i\sin\varphi$ and $z+\frac{1}{z}=2\cos\varphi$. So from the last equation, we get

$$\begin{aligned} 2i\sin 5\varphi &= 2i\sin\varphi\left(4\cos^2\varphi-4\cos^2\frac{\pi}{5}\right)\left(4\cos^2\varphi-4\cos^2\frac{2\pi}{5}\right) \\ \Rightarrow\ \sin 5\varphi &= 16\sin\varphi\left(\cos^2\varphi-\cos^2\frac{\pi}{5}\right)\left(\cos^2\varphi-\cos^2\frac{2\pi}{5}\right) \\ &= 16\sin\varphi\left(\sin^2\frac{\pi}{5}-\sin^2\varphi\right)\left(\sin^2\frac{2\pi}{5}-\sin^2\varphi\right) \end{aligned}$$

which gives

$$\sin 5\varphi=16\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}\sin\varphi\left(1-\frac{\sin^2\varphi}{\sin^2(\pi/5)}\right)\left(1-\frac{\sin^2\varphi}{\sin^2(2\pi/5)}\right)$$

Since $\sin\frac{\pi}{5}=\sqrt{\frac{5-\sqrt{5}}{8}}$, and $\sin\frac{2\pi}{5}=\sqrt{\frac{5+\sqrt{5}}{8}}$, we get $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}=\frac{\sqrt{5}}{4}$. Using this in the last equation, we finally get the required result:

$$\sin 5\varphi=5\sin\varphi\left(1-\frac{\sin^2\varphi}{\sin^2(\pi/5)}\right)\left(1-\frac{\sin^2\varphi}{\sin^2(2\pi/5)}\right)$$