Assignment

(Complex Numbers – 2)

1. Express in the form $A + i\,B$.

(i) $\dfrac{a+i\,b}{c+i\,d}$ (ii) $\dfrac{(1+i\,)^2}{1-i}$ (iii) $\left(\dfrac{1+i}{1-i}\right)^3$ (iv) $\dfrac{x+i\,y}{x-i\,y} - \dfrac{x-i\,y}{x+i\,y}$ (v) $\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$

2. Write in polar form $re^{i\varphi}$. (i) i (ii) $1+i$ (iii) -2 (iv) $-3i$ (v) $\sqrt{3}+3i$

3. Find the square roots of

(i) $8+6i$ (ii) $-1+2\sqrt{-2}$ (iii) $21-20i$ (iv) $a+i\sqrt{1-a^2}$ (v) $x+i\sqrt{x^4+x^2+1}$

(vi) $2i$ (vii) $\dfrac{2-36i}{2+3i}$ (viii) $x^2+\dfrac{1}{x^2}-\dfrac{4}{i}\left(x+\dfrac{1}{x}\right)-2$

4. If $x = 3+2i$, then show that $x^4-4x^3+4x^2+8x+39=0$. Further, if $y=\overline{x}$, then show that $x^2+xy+y^2=23$.

5. Show that the representative points of the complex numbers $1+i\,4$, $2+i\,7$, $3+i\,10$ on the Argand plane are collinear.

6. Express as the sum of two squares: (i) $(a^2+b^2)(c^2+d^2)(e^2+f^2)$ (ii) $(1+x^2)(1+y^2)(1+z^2)$.

7. If $x : y = a+i\,b : c+i\,d$, then show that

$$(c^2+d^2)x^2-2(ac+bd)xy+(a^2+b^2)y^2=0$$

8. Find the general value of θ which satisfies the equation $\displaystyle\prod_{k=1}^{n}(\cos k\theta + i\,\sin k\theta)=1$.

9. If $x+\dfrac{1}{x}=2\cos\varphi$, and $y+\dfrac{1}{y}=2\cos\theta$, prove that $\cos(\varphi+\theta)$ is one of the values of $\dfrac{1}{2}\left(xy+\dfrac{1}{xy}\right)$ and that one of the values of $\dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}$ is $2\cos(m\varphi-n\theta)$.

10. Prove that $\displaystyle\prod_{r=1}^{\infty}\left(\cos\frac{\pi}{2^r}+i\,\sin\frac{\pi}{2^r}\right)=-1$.

11. If $(a_1+i\,b_1)(a_2+i\,b_2)\cdots(a_n+i\,b_n)=A+i\,B$, prove that

(i) $\tan^{-1}\dfrac{b_1}{a_1}+\tan^{-1}\dfrac{b_2}{a_2}+\cdots+\tan^{-1}\dfrac{b_n}{a_n}=\tan^{-1}\dfrac{B}{A}$

(ii) $(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots(a_n^2+b_n^2)=A^2+B^2$

12. Find the equation whose roots are the nth powers of the roots of the equation

$$x^2-2x\cos\varphi+1=0$$

13. If $(1+x)^n=p_0+p_1x+p_2x^2+\cdots$, show that

$$p_0-p_2+p_4-\cdots=2^{n/2}\cos\tfrac{1}{4}n\pi \qquad \text{and} \qquad p_1-p_3+p_5-\cdots=2^{n/2}\sin\tfrac{1}{4}n\pi$$

14. Prove that $\sqrt[n]{a+i\,b}+\sqrt[n]{a-i\,b}$ has n real values, and find those of $\sqrt[3]{1+i\,3}+\sqrt[3]{1-i\,3}$.

15. Sketch the locus of all points in the Complex plane that satisfy: (i) $|z-2+i\,3|=2$ (ii) $|z+2\,i\,|\le 1$ (iii) $\mathrm{Re}(\overline{z}+i\,)=4$ (iv) $|z-1+2i\,|=|z+3+i\,|$ (v) $|z+i\,|+|z-i\,|=4$ (vi) $|z-i\,2|-|z+i\,2|=4$

16. If z_1 and $z_2(\neq 0)$ be two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then show that $\dfrac{z_1}{z_2}$ lies on the perpendicular bisector of the line segment joining the point $-1 + i\,0$ and $1 + i\,0$ in the Complex plane.

17. If z_1 and z_2 are two complex numbers with $|z_1| < 1$ and $|z_2| > 1$, then show that $\left|\dfrac{1 - z_1\overline{z}_2}{z_1 - z_2}\right| < 1$.

18. Let a complex number α, where $\alpha \neq 1$ be a root of the equation

$$z^{p+q} - z^p - z^q + 1 = 0$$

where p and q are distinct primes. Show that either $1+\alpha+\alpha^2+\cdots+\alpha^{p-1} = 0$ or $1+\alpha+\alpha^2+\cdots+\alpha^{q-1} = 0$ but not both together.

19. If $z = 2 + t + i\sqrt{3 - t^2}$, where $t \in \mathbb{R}$ and $t^2 < 3$, show that $\left|\dfrac{z+1}{z-1}\right|$ is independent of t. Also find the locus of z as t varies over $\mathbb{R}$.

20. Prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

and interpret the result geometrically.

21. For real numbers a_i, b_i, $i = 1, 2, \ldots, n$, prove that

$$\sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2} \leq \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2}$$

22. Find all the complex numbers z for which $\arg\left(\dfrac{3z - 6 - i\,3}{2z - 8 - i\,6}\right) = \dfrac{\pi}{4}$.

23. Solve for z: (i) $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$, (ii) $z^3 + \overline{z} = 0$.

24. Show that the roots of the equation $(z + ab)^3 = a^3$, where $|a| \neq 0$ represent the vertices of an equilateral triangle and find its area.

25. If $|z| = 1$, determine the minimum and maximum value of the expression $|1 - z| + |1 - z + z^2|$.

26. If z_1, z_2, and z_3 be the affix of a triangle in the Complex plane, find the affix of its (i) centroid, (ii) orthocenter, (iii) incenter, and (iv) circumcenter. Also find its area.

27. If z_1, z_2, z_3 be the vertices of an equilateral triangle, such that $|z_1 - a| = |z_2 - a| = |z_3 - a|$ for a complex number a, then show that $z_1 + z_2 + z_3 = 3a$.

28. Let z_1 and z_2 be the roots of the equation $z^2 + p\,z + q = 0$, where the coefficients p and q may be complex. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha(\neq 0)$ and $OA = OB$, where O is the origin, prove that

$$p^2 = 2q(1 + \cos\alpha).$$

29. For every real $a \geq 0$, find all the complex numbers z that satisfy the equation $2|z| - 4az + 1 + i\,a = 0$.

30. Show that $\displaystyle\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$.

31. Solve the equation $z^{10} - 1 = 0$ and deduce that

$$\sin 5\varphi = 5\sin\varphi\left(1 - \frac{\sin^2\varphi}{\sin^2(\pi/5)}\right)\left(1 - \frac{\sin^2\varphi}{\sin^2(2\pi/5)}\right)$$