

## Assignment

(Complex Numbers – 1)

1. Consider the complex numbers  $z_1 = 1 + 2i$ ,  $z_2 = -2 + 3i$ , and  $z_3 = 1 - i$ . Compute the following complex numbers:

- (i)  $z_1 + z_2 + z_3$   
 (ii)  $z_1 z_2 + z_2 z_3 + z_3 z_1$   
 (iii)  $z_1 z_2 z_3$   
 (iv)  $z_1^2 + z_2^2 + z_3^2$   
 (v)  $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1}$   
 (vi)  $\frac{z_1^2 + z_2^2}{z_2^2 + z_3^2}$

2. Simplify:  $E_n = \sum_{k=0}^n i^k$ .

3. Let  $z = a + ib$ . Compute  $z^2, z^3, z^4$ .

4. Compute:

- (i)  $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47}$   
 (ii)  $i^1 \cdot i^2 \cdot i^3 \dots i^{2000}$   
 (iii)  $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$

5. Find all complex numbers  $z \neq 0$  such that  $z + \frac{1}{z} \in \mathbb{R}$ .

6. Prove the following identities:

- (i)  $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2$   
 (ii)  $|1 + z_1 \bar{z}_2|^2 + |z_1 - z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$   
 (iii)  $|1 - z_1 \bar{z}_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$   
 (iv)  $|z_1 + z_2 + z_3|^2 + |-z_1 + z_2 + z_3|^2 + |z_1 - z_2 + z_3|^2 + |z_1 + z_2 - z_3|^2 = 4(|z_1|^2 + |z_2|^2 + |z_3|^2)$

7. Let  $z \neq 0$  be a complex number such that  $\left|z^3 + \frac{1}{z^3}\right| \leq 2$ . Prove that  $\left|z + \frac{1}{z}\right| \leq 2$ .

8. Find all complex numbers  $z$  such that  $|z| = 1$  and  $|z^2 + \bar{z}^2| = 1$ .

9. Find all complex numbers  $z$  such that

$$4z^2 + 8|z|^2 = 8$$

10. Let  $z_1, z_2$  be complex numbers such that  $|z_1| = |z_2| = 1$  and  $|z_1 + z_2| = \sqrt{3}$ . Compute  $|z_1 - z_2|$ .

11. Find all positive integers  $n$  such that

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + \left(\frac{-1 - i\sqrt{3}}{2}\right)^n = 2$$

12. Let  $n > 2$  be an integer. Find the number of solutions to the equation

$$z^{n-1} = i\bar{z}$$

13. Let  $z_1, z_2, z_3$  be complex numbers such that  $z_1 + z_2 + z_3 = 0$  and  $|z_1| = |z_2| = |z_3| = 1$ . Prove that  $z_1^2 + z_2^2 + z_3^2 = 0$ .

## Answers

1. (i)  $4i$  (ii)  $-4 + 5i$  (iii)  $-9 + 7i$  (iv)  $-8 - 10i$  (v)  $-\frac{311}{130} - \frac{83}{130}i$  (vi)  $\frac{152}{221} - \frac{72}{221}i$   
 2. 1 for  $n = 4m$ ,  $1 + i$  for  $n = 4m + 1$ ,  $i$  for  $n = 4m + 2$ , 0 for  $n = 4m + 3$  for  $m \in \text{Integers}$ .  
 3.  $a^2 - b^2 + 2iab, (a^3 - 3ab^2) + i(3a^2b - b^3), (a^4 + b^4 - 6a^2b^2) + i(4a^3b - 4ab^3)$   
 4. (i)  $-i$  (ii) 1 (iii)  $-3i$   
 5.  $z \in \mathbb{R}$  or  $z = x + iy$  with  $x^2 + y^2 = 1$   
 8.  $\pm\omega, \pm\omega^2, \pm i\omega, \pm i\omega^2$   
 9.  $\pm\sqrt{\frac{2}{3}}, \pm i\sqrt{2}$   
 10. 1  
 11.  $n = 3k$  for  $k \in \text{Integers}$   
 12.  $n + 1$