Assignment M–11–1

(Theory of Quadratic Equations)

1. Solve the equations (i) $3x^2-2x-1=0$, (ii) $16x-x^2=0$ (iii) $\dfrac{1}{2}x^2+x-\dfrac{1}{10}=0$, (iv) $\dfrac{2}{t}+\dfrac{2}{t+1}=1$

2. Form the equations whose roots are (i) $-\dfrac{4}{5},\ \dfrac{3}{7}$, (ii) $\dfrac{p-q}{p+q},\ -\dfrac{p+q}{p-q}$ (iii) $7\pm 2\sqrt{5}$, (iv) $-p\pm 2\sqrt{2q}$, (v) $-3\pm 5\,i$, (vi) $\pm i\,(a-b)$ (vii) $\dfrac{a}{2},\ 0,\ -\dfrac{a}{2}$, (viii) $2\pm\sqrt{3},\ 4$, (ix) $\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}$.

3. Prove that the roots of the following equations are real:

(i) $x^2-2ax+a^2-b^2-c^2=0$

(ii) $(a-b+c)x^2+4(a-b)x+(a-b-c)=0$

4. Prove that the roots of the following equations are rational:

(i) $(a+c-b)x^2+2cx+(b+c-a)=0$

(ii) $abc^2x^2+3a^2cx+b^2cx-6a^2-ab+2b^2=0$

5. Find the values of m for which the following equations will have equal roots:

(i) $x^2-15-m(2x-8)=0$

(ii) $x^2-2x(1+3m)+7(3+2m)=0$

6. For what values of m will the equation $\dfrac{x^2-bx}{ax-c}=\dfrac{m-1}{m+1}$ will have roots equal in magnitude but opposite in sign?

7. Find the value of

(i) x^3+x^2-x+22 when $x=1+2\,i$.

(ii) $x^3-3x^2-8x+15$ when $x=3+i$.

(iii) $x^3-ax^2+2a^2x+4a^3$ when $\dfrac{x}{a}=1-i\sqrt{3}$

8. If α, β are the roots of the equation $ax^2+bx+c=0$, find the values of (i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$, (ii) $\alpha^4\beta^7+\alpha^7\beta^4$, (iii) $\left(\dfrac{\alpha}{\beta}-\dfrac{\beta}{\alpha}\right)^2$, (iv) and form the equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

9. If x_1 and x_2 are the roots of $ax^2+bx+c=0$, find the value of (i) $(ax_1+b)^{-2}+(ax_2+b)^{-2}$, (ii) $(ax_1+b)^{-3}+(ax_2+b)^{-3}$.

10. Find the conditions on a, b, and c of the equation $ax^2+bx+c=0$, if (i) one root is n times the other one; (ii) opposite in sign but greater of them (in magnitude) be negative.

11. If α and β be the roots of the equation x^2+px+q, form the equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$.

12. Form the equation whose roots are the squares of the sum and difference of the roots of

$$2x^2+2(m+n)x+m^2+n^2=0$$

13. Determine the limits between which n must lie in order that the equation

$$2ax(ax+nc)+(n^2-2)c^2=0$$

may have real roots.

14. For all real values of x, show that (i) $\dfrac{x}{x^2-5x+9}$ must lie between 1 and $-\dfrac{1}{11}$, (ii) $\dfrac{x^2-x+1}{x^2+x+1}$ lies between 3 and $\dfrac{1}{3}$.

15. If α and β be the roots of the equation $x^2-px+q=0$, find the value of

(i) $\alpha^2(\alpha^2\beta^{-1}-\beta)+\beta^2(\beta^2\alpha^{-1}-\alpha)$

(ii) $(\alpha-p)^{-4}+(\beta-p)^{-4}$

16. If the roots of $\ell x^2+nx+n=0$ be in the ratio $p\ :\ q$, prove that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{\ell}}=0$$

17. If the roots of the equation $ax^2+2bx+c=0$ be α and β, and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+\delta$ and $\beta+\delta$, prove that

$$\frac{b^2-ac}{a^2}=\frac{B^2-AC}{A^2}$$

18. For what values of p will the expression $\dfrac{px^2+3x-4}{p+3x-4x^2}$ be capable of taking all real values whenever x is real?

19. Find the greatest value of $\dfrac{x+2}{2x^2+3x+6}$ for real values of x.

20. If x be real, show that the expression $\dfrac{x^2-bc}{2x-b-c}$ has no real values between b and c.

21. Prove that if the equation $ax^2+2bx+c=0$ has real and distinct roots, the equation

$$(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$$

cannot have real roots and vice versa.

22. Show that the expression $\dfrac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all real values when x is real, if a^2-b^2 and c^2-d^2 have the same sign.

23. If x be real, find the maximum value of the expression

$$2(a-x)\left(x+\sqrt{x^2+b^2}\right)$$

and indicate the value of x at which this maximum value is attained.

24. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\dfrac{pq'-p'q}{q-q'}$ or $\dfrac{q-q'}{p'-p}$.

25. If one root of the equation $x^2+ax+b=0$ is also a root of the equation $x^2+mx+n=0$, show that its other root is a root of

$$x^2+(2a-m)x+a^2-am+n=0$$

26. Find the quadratic equation whose roots are α, β (both are real) where

$$\alpha^2 + \beta^2 = 5, \qquad \text{and} \qquad 3\left(\alpha^5 + \beta^5\right) = 11\left(\alpha^3 + \beta^3\right)$$

27. Find all values of a if the equations

$$(1-2a)x^2 - 6ax - 1 = 0, \qquad \text{and} \qquad ax^2 - x + 1 = 0$$

have at least one common real root.

28. Consider the quadratic expression $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A+B$ and C are all integers. Conversely, prove that if the numbers $2A$, $A+B$ and C are integers, then $f(x)$ is an integer whenever x is an integer.

29. Solve the following equations for x:

(i) $\left(2x^2 - 3x + 1\right)\left(2x^2 + 5x + 1\right) = 9x^2$ (ii) $(x+2)(x+3)(x+8)(x+12) = 4x^2$

(iii) $(12x-1)(6x-1)(4x-1)(3x-1) = 5$ (iv) $\left(5+2\sqrt{6}\right)^{x^2-3} + \left(5-2\sqrt{6}\right)^{x^2-3} = 10$

30. If each pair of the following three equations

$$\begin{aligned} x^2 + a_1x + b_1 &= 0 \\ x^2 + a_2x + b_2 &= 0 \\ x^2 + a_3x + b_3 &= 0 \end{aligned}$$

has exactly one real root in common, show that

$$(a_1 + a_2 + a_3)^2 = 4\left(a_2a_3 + a_3a_1 + a_1a_2 - b_1 - b_2 - b_3\right)$$

31. For $a < 0$ determine all real roots of the equation $x^2 - 2a\,|x-a| - 3a^2 = 0$, where $|x|$ denotes the absolute value of x.

32. Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$ which has two distinct real roots each of which is strictly greater than zero and both of which are strictly less than unity. Find the least positive integer a for which such a quadratic equation exists.

33. Let $[x]$ denote the *greatest integer less than or equal to the real number* x. For example $[2] = 2$, $[2.7] = 2$, $[-4] = -4$, $[-2.33] = -3$ etc. If $a \in \mathbb{R}$, and the equation

$$(a-2)\left(x - [x]\right)^2 + 2\left(x - [x]\right) + a^2 = 0$$

has no integral solution and has exactly one solution that lies between 2 and 3 (excluding the end points) then show that a can be any real number lying between -1 and 0 excluding the end points themselves.

34. If $|x|$ denote the absolute value of x, solve the equation

$$2^{|x+2|} - \left|2^{x+1} - 1\right| = 2^{x+1} + 1$$

35. If p be a real parameter, find all real values of x that satisfy the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$